

NAÏVE BLOWUPS AND CANONICAL BIRATIONALLY COMMUTATIVE FACTORS

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ABSTRACT. In 2008, Rogalski and Zhang [RZ08] showed that if R is a strongly noetherian connected graded algebra over an algebraically closed field \mathbb{k} , then R has a canonical birationally commutative factor. This factor is, up to finite dimension, a twisted homogeneous coordinate ring $B(X, \mathcal{L}, \sigma)$; here X is the projective parameter scheme for point modules over R , as well as tails of points in $\text{qgr-}R$. (As usual, σ is an automorphism of X , and \mathcal{L} is a σ -ample invertible sheaf on X .)

We extend this result to a large class of noetherian (but not strongly noetherian) algebras. Specifically, let R be a noetherian connected graded \mathbb{k} -algebra, where \mathbb{k} is an uncountable algebraically closed field. Let Y_∞ denote the parameter space (or stack or proscheme) parameterizing R -point modules, and suppose there is a projective variety X that is a coarse moduli space for tails of points. There is a canonical map $p : Y_\infty \rightarrow X$. If the indeterminacy locus of p^{-1} is 0-dimensional and X satisfies a mild technical assumption, we show that there is a homomorphism $g : R \rightarrow B(X, \mathcal{L}, \sigma)$, and that $g(R)$ is, up to finite dimension, a naive blowup on X in the sense of [KRS05, RS07] and satisfies a universal property. We further show that the point space Y_∞ is noetherian.

Dedicated to Toby Stafford on the occasion of his 60th birthday, with gratitude and admiration

1. INTRODUCTION

Let \mathbb{k} be an algebraically closed field, which we assume in the introduction to be uncountable. A foundational technique of noncommutative algebraic geometry is to study a finitely generated graded \mathbb{k} -algebra $R = \mathbb{k} \oplus R_1 \oplus R_2 \oplus \cdots$ via the parameter spaces for *point modules* over R : these are cyclic (right) modules with the Hilbert series $1/(1-s)$. The main idea is due to Artin, Tate, and Van den Bergh [ATV90], although the fundamental result was proved by Rogalski and Zhang.

Rogalski and Zhang's result applies to strongly noetherian algebras. Recall that R is *strongly noetherian* if $R \otimes_{\mathbb{k}} A$ is noetherian for any commutative noetherian \mathbb{k} -algebra A . We recall also that if X is a projective scheme, $\sigma \in \text{Aut}_{\mathbb{k}}(X)$, and \mathcal{L} is an invertible sheaf on X , then we may define a *twisted homogeneous coordinate ring* $B(X, \mathcal{L}, \sigma)$ by

$$B(X, \mathcal{L}, \sigma) := \bigoplus_{n \geq 0} H^0(X, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}).$$

(Here, as usual, we define $\mathcal{L}^\sigma := \sigma^* \mathcal{L}$.) If \mathcal{L} is appropriately ample (the technical term is σ -*ample*, defined in Section 7), then $B(X, \mathcal{L}, \sigma)$ is noetherian by [AV90, Theorem 1.4], and is strongly noetherian by [ASZ99, Proposition 4.13].

Rogalski and Zhang's result is:

Theorem 1.1. ([RZ08, Theorem 1.1]) *Let $R = \mathbb{k} \oplus R_1 \oplus R_2 \oplus \cdots$ be a strongly noetherian graded algebra, generated in degree 1. Then there is a \mathbb{k} -algebra homomorphism $g : R \rightarrow B(X, \mathcal{L}, \sigma)$, where X is a projective scheme, $\sigma \in \text{Aut}_{\mathbb{k}}(X)$, and \mathcal{L} is a σ -ample invertible sheaf on X . Further, g is surjective in large degree and satisfies a universal property.*

The universal property in the theorem means, roughly, that the kernel of g annihilates any point module. We prove that g satisfies a different universal property in Theorem 4.12.

The projective scheme X in Theorem 1.1 is the parameter scheme for point modules; it is a consequence of R being strongly noetherian that point modules are parameterized by a scheme rather than a more exotic object [AZ01, Corollary E4.5]. If R is a (3-dimensional) Sklyanin algebra, then X is an elliptic curve and we recover the well-known embedding [ATV90] of an elliptic curve in a noncommutative \mathbb{P}^2 .

Ring theory would be less interesting if all noetherian algebras were strongly noetherian. Fortunately, this is not true, as shown by Rogalski [Rog04]. For example, let X be a variety of dimension ≥ 2 and

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let σ, \mathcal{L} be as above. Let $P \in X$, and assume that the σ -orbit of P *critically dense*: that is, it is infinite, and any infinite subset is Zariski-dense in X . (More generally, we may take P to be any 0-dimensional subscheme of X supported at points with critically dense orbits.) We may use these data to construct a subring

$$R = R(X, \mathcal{L}, \sigma, P) \subseteq B(X, \mathcal{L}, \sigma),$$

by setting $R_n := H^0(X, \mathcal{I}_P \mathcal{L} (\mathcal{I}_P \mathcal{L})^\sigma \cdots (\mathcal{I}_P \mathcal{L})^{\sigma^{n-1}})$. The ring $R(X, \mathcal{L}, \sigma, P)$ is called the *naïve blowup algebra* associated to the data $(X, \mathcal{L}, \sigma, P)$. By [KRS05, Theorem 1.1] and [RS07, Theorem 1.1], $R(X, \mathcal{L}, \sigma, P)$ is always noetherian but never strongly noetherian, and thus Theorem 1.1 does not apply. However, there is trivially a map from $R(X, \mathcal{L}, \sigma, P)$ to the overring $B(X, \mathcal{L}, \sigma)$. It is a bit embarrassing that current theory does not recover this inclusion.

In this paper we remedy the embarrassment. Our results apply, however, to a much larger class of rings: to all those whose point modules have the geometry of a naïve blowup algebra. Let S be a noetherian graded algebra, generated in degree 1. We seek to answer two questions. First, is there a canonical map from S to a twisted homogeneous coordinate ring? Second, what can be said about the image of S under this map?

To describe our main theorem, we establish notation for functors of points over noncommutative rings. Let $R = \mathbb{k} \oplus R_1 \oplus \cdots$ be a finitely generated graded algebra generated in degree 1. We will denote the functor of (embedded) R -point modules by F . That is, if A is a commutative \mathbb{k} -algebra, then $F(A)$ is the set of graded cyclic right $R \otimes_{\mathbb{k}} A$ -modules M such that each M_n is a rank 1 projective A -module. Let M, N be R -point modules. We say that $M \sim N$ if $M_{\geq n} \cong N_{\geq n}$ for $n \gg 0$; that is, M and N are equal as objects of noncommutative projective geometry, or, more formally, of the category $\text{qgr-}R$ of “tails” of graded modules (see [AZ01]). Let $G := F/\sim$, and let $\pi : F \rightarrow G$ be the quotient map.

If we are willing to move beyond the category of schemes, the functor F is always represented by a geometric object, which we call Y_∞ , and refer to as the *point space* of R ; see Section 2. The geometry of G is often more complicated. For example, let $R(X, \mathcal{L}, \sigma, P)$ be a naïve blowup algebra as above. In an earlier paper [NS10, Theorem 1.3], the authors proved that in this situation in fact there is a morphism $G \rightarrow X$ that makes X a coarse moduli space for G , so X *corepresents* G . In contrast, by [KRS05, Theorem 1.1], G is not represented by any scheme of finite type.

Our main result may be thought of as a converse of sorts to [NS10, Theorem 1.3]. We show that, under a restriction on the geometry of point modules, the existence of a coarse moduli scheme X for G ensures a canonical map from R to a twisted homogeneous coordinate ring on X . Further, we show that the image of R is (up to finite dimension) a naïve blowup algebra. More specifically, we have:

Theorem 1.2. *Let \mathbb{k} be an uncountable algebraically closed field, and let $R = \mathbb{k} \oplus R_1 \oplus \cdots$ be a noetherian graded \mathbb{k} -algebra generated in degree 1. Let Y_∞ be the point space of R , as above. Suppose the following:*

- (i) *there is a commutative diagram of morphisms*

$$\begin{array}{ccc} & F \cong Y_\infty & \\ \pi \swarrow & & \searrow p \\ G & \xrightarrow{\quad} & X, \end{array}$$

where X is a projective scheme and a coarse moduli space for G .

Suppose further that:

- (ii) *X is a variety of dimension ≥ 2 that is either a surface or locally factorial at all points in the indeterminacy locus of p^{-1} ;*
- (iii) *the map $G \rightarrow X$ is bijective on \mathbb{k} -points;*
- (iv) *the indeterminacy locus of p^{-1} consists (set-theoretically) of countably many points.*

Then:

- (1) *there are an automorphism σ of X , a σ -ample invertible sheaf \mathcal{L} on X , and a \mathbb{k} -algebra homomorphism $g : R \rightarrow B(X, \mathcal{L}, \sigma)$, universal for maps from R to birationally commutative algebras;*
- (2) *there is a 0-dimensional subscheme P of X , supported at points with critically dense orbits, so that the image of g is equal (in large degree) to the naïve blowup algebra $R(X, \mathcal{L}, \sigma, P)$.*

We further obtain:

- (3) *the point space Y_∞ is noetherian; and*
- (4) *the indeterminacy locus of p^{-1} is critically dense in X .*

See Theorem 8.1 for a more detailed and technically precise statement of this result.

We show in Corollary 3.5 that the hypotheses of Theorem 1.2 hold if $R = R(X, \mathcal{L}, \sigma, P)$ is itself a naïve blowup algebra as above. Our result thus recovers the inclusion of $R(X, \mathcal{L}, \sigma, P)$ in $B(X, \mathcal{L}, \sigma)$.

The universal property in the theorem means, approximately, that any map from R to a skew polynomial extension of a commutative \mathbb{k} -algebra factors through g . The existence of a factor of R universal for maps from R to birationally commutative algebras holds in substantial generality (see Theorem 4.9) and does not require any assumptions on the geometry of the point space Y_∞ .

In [NS10] we drew a parallel between Y_∞ and the Hilbert scheme $S^{[n]}$ of n points on S . We claimed there that Y_∞ should be thought of as a “Hilbert scheme of one point on a noncommutative variety” and that X should be thought of as a “coarse moduli space of one point,” analogous to the symmetric product $\mathrm{Sym}^n(S)$. In light of this analogy, it is perhaps not surprising that we are not certain whether condition (iii) of the theorem, although it appears highly plausible, is automatically satisfied for reasonable classes of algebras. Indeed, the analogue for moduli of zero-dimensional sheaves on a smooth commutative surface fails: it is well-known that $S^{[n]} \rightarrow \mathrm{Sym}^n(S)$ is a (nontrivial) resolution of singularities for $n > 1$.

We briefly describe the structure of the paper. In Section 2 we establish notation for point spaces of algebras and define the point space Y_∞ rigorously. In Section 3, we describe the point space of a naïve blowup algebra and show that the hypotheses of Theorem 1.2 apply. In Section 4 we show, under very weak conditions, that the point space gives rise to a factor of R universal for maps to birationally commutative algebras. We show that graded ideals I of R so that R/I is birationally commutative correspond to certain substacks of Y_∞ , which we call “Stafford stacks.” We believe these results are of independent interest. In Section 5 we assume the existence of a coarse moduli scheme X for G . We construct both the automorphism σ of X and a map $g : R \rightarrow \mathbb{k}(X)[t; \sigma]$. In Section 6 we study how points and curves in X lift to Y_∞ . We prove the key result that any point where p^{-1} is not defined must have infinite order under σ . In Section 7 we define the rest of the data for the naïve blowup algebra $R(X, \mathcal{L}, \sigma, P)$. We show that \mathcal{L} is invertible and σ -ample, and that the orbits of points in P are dense. Finally, in Section 8 we prove Theorem 1.2.

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Notation. The symbol \mathbb{k} will always denote an algebraically closed field, often but not always uncountable. All algebras and schemes will be defined over \mathbb{k} . Modules are by default right modules.

2. GENERALITIES

In this section, we place the geometry of point schemes on a firmer footing and define the spaces Y_∞ rigorously.

A *connected graded* \mathbb{k} -algebra is an \mathbb{N} -graded \mathbb{k} -algebra $R = R_0 \oplus R_1 \oplus \cdots$, with $R_0 = \mathbb{k}$ and all R_i finite-dimensional. Let R be a connected graded \mathbb{k} -algebra generated in degree 1. We use subscript notation to denote base extension; that is, if A is a commutative \mathbb{k} -algebra, let $R_A := R \otimes_{\mathbb{k}} A$. Recall that a graded R -module M is *bounded* if there is an n_0 such that $M_n = 0$ for $n \geq n_0$. If all finitely generated submodules of a module M are bounded, we say that M is *torsion*. If M has no nonzero bounded submodules, then M is *torsion-free*.

An R_A -*point module* is a graded quotient M of R_A so that M_i is rank 1 projective over A for all $i \geq 0$; equivalently, M is a graded cyclic R_A -module, flat over A , with Hilbert series $1/(1-s)$. There is a (covariant) functor of points $F : \text{Commutative } \mathbb{k}\text{-algebras} \rightarrow \text{Sets}$ given by $F(A) = \{ R_A\text{-point modules} \}$. We sometimes, without comment, treat F as a contravariant functor from (affine) Schemes to Sets.

The functor F is a Hilbert functor as in [AZ01], associated to the Hilbert series $1/(1-s)$. If we consider the equivalent functor for the Hilbert series $1 + s + \cdots + s^n$, it is clear that it is represented by a projective scheme. Let this scheme be Y_n (for $n \in \mathbb{N}$). That is, Y_n parameterizes cyclic R -modules $M = M_0 \oplus \cdots \oplus M_n$, where $\dim M_i = 1$ for $0 \leq i \leq n$. Note that $Y_0 = \mathrm{Spec} \mathbb{k}$.

The truncation map $M \mapsto M/M_n$ gives a morphism $\phi_n : Y_n \rightarrow Y_{n-1}$, and the functor of points F is represented in abstract terms by $Y_\infty := \varprojlim_{\phi_n} Y_n$. As in [NS10], we consider Y_∞ as a stack. In this language, a morphism from a scheme W to Y_∞ is given by a collection of compatible maps $W \rightarrow Y_n$. A

morphism from Y_∞ to W is defined as a morphism of functors from Y_∞ to h_W , the functor of points of W . For example, there is a natural morphism $\Phi_n : Y_\infty \rightarrow Y_n$ that sends a point module M to $M/M_{\geq n+1}$.

As in [NS10, Proposition 4.2], the space Y_∞ may be formally given the structure of an *fpqc-algebraic stack* (this nonstandard terminology is defined in [NS10], Section 4). We may also consider Y_∞ as a proscheme; this is the perspective of [AZ01]. However, these two points of view are equivalent. This follows from:

Proposition 2.1. *Let $\cdots Y_{n+1} \rightarrow Y_n \rightarrow \cdots \rightarrow Y_1$ be a system of projective \mathbb{k} -schemes and scheme-theoretically surjective morphisms (that is, morphisms $Y_{n+1} \rightarrow Y_n$ with scheme-theoretic image Y_n). Let $Y_\infty := \varprojlim Y_n$ be the fpqc-algebraic stack defined as above. Suppose $p : Y_\infty \rightarrow X$ is a morphism to a scheme X of finite type over \mathbb{k} . Then there is a compatible system of factorizations of the morphism p through the schemes Y_n for all n sufficiently large.*

Proof. Fix $n_0 \geq 1$. Choose a finite cover of Y_{n_0} by affine schemes $U_{n_0, \alpha}$ and let $U_{n_0} := \coprod U_{n_0, \alpha}$. Now, given a cover $\{U_{n, \alpha}\}$ of Y_n by affines, let $\{U_{n+1, \alpha}\}$ be a finite affine cover subordinate to the cover of Y_{n+1} by the inverse images of the $U_{n, \alpha}$; then, defining $U_{n+1} = \coprod U_{n+1, \alpha}$, we get a commutative diagram

$$\begin{array}{ccc} Y_{n+1} & \longleftarrow & U_{n+1} \\ \downarrow & & \downarrow \\ Y_n & \longleftarrow & U_n \end{array}$$

with scheme-theoretically surjective vertical arrows. Since each U_n is a finite disjoint union of affine schemes it is itself affine, say $U_n = \text{Spec}(T_n)$, and since $U_{n+1} \rightarrow U_n$ is scheme-theoretically surjective we get a system of injective ring homomorphisms $T_n \hookrightarrow T_{n+1} \hookrightarrow \cdots$. Let $T := \varinjlim T_n$ and $U_\infty := \text{Spec}(T)$. Then, since $U_\infty = \varprojlim U_n$, we get a morphism $U_\infty \rightarrow Y_\infty$ making all the squares

$$\begin{array}{ccc} Y_\infty & \longleftarrow & U_\infty \\ \downarrow & & \downarrow \\ Y_n & \longleftarrow & U_n \end{array}$$

commute.

Since X is of finite type, the composite $U_\infty \rightarrow Y_\infty \rightarrow X$ factors through a finite-type affine scheme $W = \text{Spec } T'$ for some finite-type subalgebra T' of T . As T is the union of the T_n , we have $T' \subseteq T_n$ for $n \gg 0$. It follows that $U_\infty \rightarrow X$ factors through morphisms $U_n \rightarrow X$ for all n sufficiently large.

We now use (Zariski) descent to obtain a morphism $Y_n \rightarrow X$ for $n \gg 0$. To show that $U_n \rightarrow X$ is the composite $U_n \rightarrow Y_n \rightarrow X$ with a morphism $Y_n \rightarrow X$, it suffices to check that the composites $U_n \times_{Y_n} U_n \rightarrow U_n \rightarrow X$ with the two projections $\Pi_i : U_n \times_{Y_n} U_n \rightarrow U_n$ ($i = 1, 2$) coincide. By hypothesis each Y_n is projective, hence separated. It follows that each fiber product $U_n \times_{Y_n} U_n$ is again an affine scheme, say $U_n \times_{Y_n} U_n = \text{Spec}(\tilde{T}_n)$. Moreover, the map $U_{n+1} \times_{Y_{n+1}} U_{n+1} \rightarrow U_n \times_{Y_n} U_n$ is scheme-theoretically surjective for each n (each of these is a disjoint union of intersections of the open sets in the affine open covers of Y_{n+1} , respectively Y_n , so this follows from the scheme-theoretic surjectivity of $Y_{n+1} \rightarrow Y_n$ since the cover of Y_{n+1} refines the cover of Y_n).

Observe next that $\varprojlim U_n \times_{Y_n} U_n = U_\infty \times_{Y_\infty} U_\infty$ (cf. Lemma 4.5 of [NS10]). Because the map $U_\infty \rightarrow X$ is defined as the composite $U_\infty \rightarrow Y_\infty \rightarrow X$, the two maps $U_\infty \times_{Y_\infty} U_\infty \xrightarrow{\Pi_i} U_\infty \rightarrow X$, where Π_i the projection on the i th factor, coincide. But we have:

Lemma 2.2. *Suppose $f, g : Z \rightrightarrows X$ are two morphisms of schemes and $\Pi : Z' \rightarrow Z$ is scheme-theoretically surjective. Then $f = g$ if and only if $f\Pi = g\Pi$.*

Proof of Lemma. By definition, Π is surjective as a map of topological spaces. Hence $f = g$ as maps of topological spaces if and only if $f\Pi = g\Pi$ as maps of topological spaces. Now suppose that $f\Pi = g\Pi$ as morphisms of schemes; thus $f = g$ as maps of topological spaces. We thus have $(f\Pi)_* \mathcal{O}_{Z'} = (g\Pi)_* \mathcal{O}_{Z'}$. Scheme-theoretic surjectivity of Π means that $\mathcal{O}_Z \rightarrow \Pi_* \mathcal{O}_{Z'}$ is injective; thus, so are $f_* \mathcal{O}_Z \rightarrow (f\Pi)_* \mathcal{O}_{Z'}$ and $g_* \mathcal{O}_Z \rightarrow (g\Pi)_* \mathcal{O}_{Z'} = (f\Pi)_* \mathcal{O}_{Z'}$. It follows that $f_* \mathcal{O}_Z = g_* \mathcal{O}_Z$ and that

$$\mathcal{O}_X \xrightarrow{f-g} f_* \mathcal{O}_Z \text{ is zero iff } \mathcal{O}_X \xrightarrow{f-g} f_* \mathcal{O}_Z \rightarrow (f\Pi)_* \mathcal{O}_{Z'} \text{ is zero,}$$

completing the proof. □

Since $U_\infty \times_{Y_\infty} U_\infty \rightarrow U_n \times_{Y_n} U_n$ is an inverse limit of scheme-theoretically surjective maps of affine schemes, and hence is itself scheme-theoretically surjective, it follows that the composite maps $U_n \times_{Y_n} U_n \xrightarrow{\Pi_i} U_n \rightarrow X$ coincide. So the map $U_n \rightarrow X$ descends to a map $Y_n \rightarrow X$, as desired. \square

Corollary 2.3. *Let $\cdots Y_{n+1} \xrightarrow{\phi_{n+1}^+} Y_n \rightarrow \cdots \rightarrow Y_1$ and $\cdots X_{n+1} \rightarrow X_n \rightarrow \cdots \rightarrow X_1$ be systems of projective \mathbb{k} -schemes and morphisms. Let $Y_\infty = \varprojlim Y_j$ and $X_\infty = \varprojlim X_i$ be the fpqc-algebraic stacks defined above. Let Y'_j be the (scheme-theoretic) image of the natural map from $Y_\infty \rightarrow Y_j$. Then*

$$\mathrm{Hom}(Y_\infty, X_\infty) = \varprojlim_i \varinjlim_j \mathrm{Hom}(Y'_j, X_i).$$

Proof. By definition,

$$\mathrm{Hom}(Y_\infty, X_\infty) = \varprojlim_i \mathrm{Hom}(Y_\infty, X_i).$$

Thus it suffices to prove the corollary in the case that $X_\infty = X_i$ is a projective scheme.

For all $n \geq m \in \mathbb{N}$, we write $\phi^{n-m} : Y_n \rightarrow Y_m$ for the induced morphism. Now, $Y'_m = \bigcap_j \phi^j(Y_{m+j})$ is easily seen to be a closed subscheme of Y_m and is therefore projective; further, $Y_\infty = \varprojlim Y'_m$. That is, we may replace Y_m by Y'_m and assume without loss of generality that the ϕ^k are all scheme-theoretically surjective. But now it follows directly from Proposition 2.1 that $\mathrm{Hom}(Y_\infty, X_i) = \varinjlim_j \mathrm{Hom}(Y'_j, X_i)$. \square

Note that

$$\mathrm{Hom}_{\mathrm{proscheme}}(Y_\infty, X_\infty) = \varprojlim_i \varinjlim_j \mathrm{Hom}(Y_j, X_i)$$

(see [AM86, Definition A.2.1]) Thus Corollary 2.3 implies that we may equivalently think of X_∞ and Y_∞ as pro-schemes or as fpqc-algebraic stacks, as long as we are willing to replace Y_j by Y'_j .

In the sequel, we will continue to speak of the geometry on an object such as Y_∞ informally, considering it rigorously as either a stack or as a pro-scheme only if necessary.

We will always use the following notation for parameter spaces of point modules.

Notation 2.4. Let R be a connected graded \mathbb{k} -algebra generated in degree 1. Let Y_n (for $n \in \mathbb{N}$) be the projective scheme representing graded cyclic R -modules with Hilbert series $1 + s + \cdots + s^n$.

If $n \geq 1$, there are two morphisms $\phi_n, \psi_n : Y_n \rightarrow Y_{n-1}$, where $\phi_n(M) = M/M_n$ and $\psi_n(M) = M[1]_{\geq 0}$. Clearly $\phi_{n-1}\psi_n = \psi_{n-1}\phi_n$. We will write $\phi^k : Y_{n+k} \rightarrow Y_n$ to denote the map $\phi_{n+1}\phi_{n+2} \cdots \phi_{n+k}$, and similarly for ψ^k . By $\phi^k(Y_{n+k})$ we mean the scheme-theoretic image of $\phi^k : Y_{n+k} \rightarrow Y_n$, and similarly for ψ^k .

Let $Y_\infty := \varprojlim_\phi Y_n$. We refer to Y_∞ as the *point space* of R , and to the collection $\{Y_n, \phi_n, \psi_n\}$ as the *point scheme data* of R . Let $\Psi : Y_\infty \rightarrow Y_\infty$ be the endomorphism of Y_∞ induced from the maps ψ_n ; that is, $\Psi(M) = M[1]_{\geq 0}$. Let $\Phi_n : Y_\infty \rightarrow Y_n$ be the canonical morphism.

Let $F \cong Y_\infty$ be the functor of embedded point modules of R . We note that Y_∞ represents F . Let G be the functor of point modules up to isomorphism in $\mathrm{qgr}\text{-}R$. That is, define $M \sim N$ if $M_{\geq k} \cong N_{\geq k}$ for $k \gg 0$; equivalently, if $\Psi^k(M) \cong \Psi^k(N)$. Then G is the sheafification (in the fpqc topology) of F/\sim . There is a canonical quotient morphism $\pi : F \rightarrow G$.

If $\delta : W \rightarrow Y_\infty$ is a morphism, we let $\delta_n := \Phi_n \delta : W \rightarrow Y_n$ without comment.

We will often use the following result of Rogalski and Stafford.

Proposition 2.5. ([RS09, Corollary 2.7]) *Fix $N \in \mathbb{N}$. Let R be a noetherian connected graded \mathbb{k} -algebra generated in degree 1, and adopt Notation 2.4. For each $n \geq N$, let H_n be a finite set of (not necessarily closed) points of Y_n , such that $\phi_n(H_n) \subseteq H_{n-1}$, or respectively $\psi_n(H_n) \subseteq H_{n-1}$. Then for $n \gg N$, the cardinality $\#H_n$ is constant, and ϕ_n , or respectively ψ_n , gives a bijection between H_n and H_{n-1} . Moreover, if $x \in H_{n-1}$ for any such n , then ϕ_n^{-1} , or respectively ψ_n^{-1} , is defined and is a local isomorphism at x .* \square

3. POINT SPACES OF NAÏVE BLOWUPS

In this section, we apply the constructions in Section 2 to naïve blowup algebras. We show that a naïve blowup algebra that is generated in degree 1 gives an example of the geometry in Theorem 1.2. Most results here are immediate consequences of results in [NS10].

We will need to work briefly with bimodule algebras (see [Van96]). Here we give only a few definitions; more detail appears in [NS10]. Let X be a scheme. An \mathcal{O}_X -bimodule is a quasicoherent sheaf \mathcal{F} on $X \times X$,

with a left structure given by $(\mathrm{pr}_1)_*\mathcal{F}$ and a right structure given by $(\mathrm{pr}_2)_*\mathcal{F}$. The basic example is the bimodule

$$\mathcal{G}_\sigma := (1 \times \sigma)_*\mathcal{G},$$

where \mathcal{G} is a quasicoherent sheaf on X and $\sigma \in \mathrm{Aut}_{\mathbb{k}}(X)$. Note that $\mathcal{F}_\sigma \otimes \mathcal{G}_\tau \cong (\mathcal{F} \otimes_X \mathcal{G}^\sigma)_{\tau\sigma}$ by [KRS05, Lemma 2.3]. If \mathcal{G} is a quasicoherent sheaf on X it may be regarded trivially as a bimodule via the diagonal map $(1 \times 1)_*\mathcal{G}$, and we will do so without comment. A *bimodule algebra* is an algebra object in the category of bimodules: that is, a bimodule \mathcal{R} with an associative multiplication $\mathcal{R} \otimes \mathcal{R} \rightarrow \mathcal{R}$.

Let X be a projective variety of dimension ≥ 2 . Let $\sigma \in \mathrm{Aut}(X)$ and let \mathcal{L} be a σ -ample invertible sheaf on X . As usual, we define $\mathcal{L}_n := \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$. Let P be a zero-dimensional subscheme of X . We define a bimodule algebra $\mathcal{R} = \mathcal{R}(X, \mathcal{L}, \sigma, P)$ as follows: we have $\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{R}_n$, where

$$\mathcal{R}_n = \mathcal{I}_P \mathcal{I}_P^\sigma \cdots \mathcal{I}_P^{\sigma^{n-1}} \mathcal{L}_n.$$

Then \mathcal{R} has a natural multiplicative structure induced by the multiplication maps

$$\mathcal{R}_n \otimes \mathcal{R}_m^{\sigma^n} \rightarrow \mathcal{R}_{n+m}.$$

Note that $R(X, \mathcal{L}, \sigma, P) = H^0(X, \mathcal{R})$.

There is a natural way to define a (right) module over the bimodule algebra \mathcal{R} . We will only need this in the following specific case:

Definition 3.1. Let $\mathcal{R} := \mathcal{R}(X, \mathcal{L}, \sigma, P)$. The graded quasi-coherent sheaf $\mathcal{M} := \bigoplus \mathcal{M}_n$ on X is a *right \mathcal{R} -module* if there are maps $\mathcal{M}_n \otimes \mathcal{R}_m^{\sigma^n} \rightarrow \mathcal{M}_{n+m}$ for all $n, m \in \mathbb{N}$ that satisfy the appropriate associativity condition.

For example, if \mathcal{I} is an ideal sheaf on X , then $\mathcal{I}\mathcal{R}$ is a right \mathcal{R} -submodule of \mathcal{R} .

The algebra $R(X, \mathcal{L}, \sigma, P)$ and the bimodule algebra $\mathcal{R}(X, \mathcal{L}, \sigma, P)$ may be defined for any 0-dimensional P , and in the sequel we will do so. In this section, however, we will assume that P is supported on points with critically dense orbits. Then $\mathcal{R}(X, \mathcal{L}, \sigma, P)$ and $R(X, \mathcal{L}, \sigma, P)$ are noetherian [RS07, Proposition 2.12, Theorem 3.1]. As in [NS10], for any $\ell \in \mathbb{N}$ we may define a noetherian fpqc-algebraic stack ${}_\ell Z_\infty$, parameterising ℓ -shifted point modules over \mathcal{R} . A \mathbb{k} -point of ${}_\ell Z_\infty$ is a graded factor \mathcal{M} of the right \mathcal{R} -module $\mathcal{R}_{\geq \ell}$ so that there is some $x \in X$ with $\mathcal{M}_n \cong \mathbb{k}_x$ for $n \geq \ell$. There is a natural morphism

$$\begin{aligned} r : {}_\ell Z_\infty &\rightarrow X \\ \mathcal{M} &\mapsto \mathrm{Supp} \mathcal{M}. \end{aligned}$$

The map r is easy to understand; in particular, we immediately have:

Lemma 3.2. *Let $X, \mathcal{L}, \sigma, P$ as above, and let $\ell \in \mathbb{N}$. Let $r : {}_\ell Z_\infty \rightarrow X$ be the morphism defined above. Then the indeterminacy locus of r^{-1} is precisely*

$$\Omega := \bigcup \{ \sigma^{-n}(\mathrm{Supp} P) \mid n \geq 0 \}.$$

Proof. Let $x \in X \setminus \Omega$. All \mathcal{R}_n are invertible at x . It is clear that $\mathcal{R}_{\geq \ell} / \mathcal{I}_x \mathcal{R}_{\geq \ell}$ is the unique point in $r^{-1}(x)$. A similar argument shows that the scheme-theoretic fiber of x is also trivial. Thus the indeterminacy locus of r^{-1} is contained in Ω .

For $n \in \mathbb{N}$, let $\mathcal{I}_n := \mathcal{I}_P \cdots \mathcal{I}_P^{\sigma^{n-1}}$. The first paragraph shows that r^{-1} is defined at the generic point of X . By [NS10, Theorem 5.11], there is a connected component X_∞ of ${}_\ell Z_\infty$ so that $r|_{X_\infty} : X_\infty \rightarrow X$ is birational, and so that

$$X_\infty \cong \varprojlim \mathrm{Bl}_{\mathcal{I}_n} X.$$

The indeterminacy locus of the inverse birational map $X \dashrightarrow X_\infty$ is Ω , and so the indeterminacy locus of r^{-1} is precisely Ω . \square

Let $\ell \in \mathbb{N}$. As in [NS10], let ${}_\ell Y_\infty$ be the fpqc-algebraic stack which parameterises ℓ -shifted point modules of \mathcal{R} : that is, factors of $\mathcal{R}_{\geq \ell}$ with Hilbert series $t^\ell / (1 - t)$. For $\ell \gg 0$, the stacks ${}_\ell Z_\infty$ and ${}_\ell Y_\infty$ are isomorphic, under a mild technical hypothesis.

Theorem 3.3. (cf. [NS10, Theorem 6.8]) *Let $X, \mathcal{L}, \sigma, P$ be as above, and let $R := R(X, \mathcal{L}, \sigma, P)$.*

- (1) *Suppose that $R_n = (R_1)^n$ for $n \gg 0$. Then there is $\ell_0 \in \mathbb{N}$ so that for $\ell \geq \ell_0$, the global sections functor induces an isomorphism*

$$s : {}_\ell Z_\infty \rightarrow {}_\ell Y_\infty.$$

The inverse morphism $s^{-1} : {}_\ell Y_\infty \rightarrow {}_\ell Z_\infty$ acts as follows on \mathbb{k} -points. Let $y \in {}_\ell Y_\infty$ be represented by a graded module $S_{\geq \ell}/J$. Then $s^{-1}(y)$ is represented by the factor $\mathcal{R}_{\geq \ell}/\mathcal{J}$, where \mathcal{J}_n is the subsheaf of \mathcal{R}_n generated by the sections in J_n .

- (2) If we replace R by a sufficiently large Veronese, or alternatively replace \mathcal{L} by a sufficiently ample \mathcal{L}' , then R itself is generated in degree 1 and we may take $\ell_0 = 0$ above.

Proof. (1). If R is generated in degree 1, this is [NS10, Theorem 6.8]. The statement here is an immediate generalisation.

(2). By [RS07, Proposition 3.18], $R^{(n)}$ is generated in degree 1 for all $n \gg 0$. We then apply (1) to $R^{(n)}$; if we replace $R^{(n)}$ by $R^{(n\ell_0)}$ then clearly we do not need to shift the point modules.

By [RS07, Proposition 3.20], if \mathcal{M} is ample and globally generated then some $R(X, \mathcal{M}^{\otimes m}, \sigma, P)$ is generated in degree 1. That we may also take $\ell_0 = 0$ (possibly by changing the invertible sheaf again) is the comment after Definition 6.3 of [NS10]. \square

In the next result, we use the isomorphism s of Theorem 3.3 to study the unshifted point schemes of a naïve blowup algebra. We number the parts of this result to be consistent with Theorem 1.2.

Proposition 3.4. *Let $X, \mathcal{L}, \sigma, P$ be as above and let $R := R(X, \mathcal{L}, \sigma, P)$. Suppose that $R_n = R_1^n$ for $n \gg 0$.*

Let S be a graded subalgebra of R so that S is generated in degree 1 and $S_n = R_n$ for $n \gg 0$. Let Y_∞ be the point space for S , and let $\Psi : Y_\infty \rightarrow Y_\infty$ be the shift morphism $M \mapsto M[1]_{\geq 0}$.

Then there is a morphism $q : Y_\infty \rightarrow X$ that satisfies the following properties:

- (i) *the morphism q is constant on \sim -equivalence classes, and the factor morphism*

$$\begin{array}{ccc} & F \cong Y_\infty & \\ \pi \swarrow & & \searrow q \\ G & \xrightarrow{\quad} & X \end{array}$$

makes X a coarse moduli space for q ;

- (iii) *the map $G \rightarrow X$ is bijective on \mathbb{k} -points;*

- (iv) *the indeterminacy locus of q^{-1} is $\bigcup \{\sigma^{-n}(\text{Supp } P) \mid n \geq 0\}$, and in particular is countable.*

Furthermore,

- (*) *Y_∞ is noetherian; and*

- (†) *$q\Psi = \sigma q$.*

Proof. Let $\ell_0 \in \mathbb{N}$ be given by Theorem 3.3(1), and let $\ell \geq \ell_0$ be such that $S_{\geq \ell} = R_{\geq \ell}$. Then ${}_\ell Y_\infty$ also parameterises ℓ -shifted point modules over R .

As in the proof of [NS10, Theorem 6.8], there is a natural “tail” morphism

$$T : Y_\infty \rightarrow {}_\ell Y_\infty$$

given by restricting a surjection $f : S \rightarrow M$ to $f_{\geq \ell} : S_{\geq \ell} \rightarrow M_{\geq \ell}$. Let $q : Y_\infty \rightarrow X$ be the composition

$$\begin{array}{ccccc} Y_\infty & \xrightarrow{T} & {}_\ell Y_\infty & \xrightarrow{s^{-1}} & {}_\ell Z_\infty \\ & \searrow q & & & \downarrow r \\ & & & & X. \end{array}$$

We verify that the conclusions of the proposition hold for q .

- (iv). By Lemma 3.2, the indeterminacy locus of r^{-1} is

$$\Omega := \bigcup \{\sigma^{-n}(\text{Supp } P) \mid n \geq 0\}.$$

We claim that this is also the indeterminacy locus of q^{-1} .

Let $Y_\infty^{(\ell)}$ be the point space for the Veronese $R^{(\ell)} = S^{(\ell)}$. To give an ℓ -shifted point module M over S , it is equivalent to give a right ideal J of S so that $J_{< \ell} = 0$ and $J_{\geq \ell}$ is the kernel of the surjection $S_{\geq \ell} \rightarrow M$. Let $J^{(\ell)} := \bigoplus_n J_{n\ell}$. Then there is a natural morphism $v : {}_\ell Y_\infty \rightarrow Y_\infty^{(\ell)}$, where $v(M) = S^{(\ell)}/J^{(\ell)}$.

The composition $v \circ T : Y_\infty \rightarrow Y_\infty^{(\ell)}$ sends $M \mapsto M^{(\ell)}$. By [RS07, Lemma 3.7], $v \circ T$ is a closed immersion. Thus T is a closed immersion and so therefore is $s^{-1}T$. Thus to show that Ω is the indeterminacy locus of q^{-1} , it suffices to show that $\text{Im}(q) \supseteq X \setminus \Omega$.

Let $\mathcal{R} := \mathcal{R}(X, \mathcal{L}, \sigma, P)$. Let $x \in X \setminus \Omega$ be a \mathbb{k} -point. Define

$$J_x := \bigoplus_{n \geq 0} S_n \cap H^0(X, \mathcal{I}_x \mathcal{R}_n).$$

The codimension of $(J_x)_n$ in S_n is 0 or 1 for all n . However, it follows from the fact that $R_n = (S_1)^n$ for $n \gg 0$ that the sections in S_n generate \mathcal{R}_n for all $n \in \mathbb{N}$. Thus we cannot have $S_n = (J_x)_n$, and $M_x := S/J_x$ is a point module. From the construction of s^{-1} in Theorem 3.3, we see that $q(M_x) = x$.

(i). By construction, q does not depend on the particular value of ℓ chosen, as long as ℓ is sufficiently large. It follows that q is constant on \sim -equivalence classes and thus factors through G . We thus have a commutative diagram

$$\begin{array}{ccc} Y_\infty & \xrightarrow{s^{-1}T} & {}_\ell Z_\infty \\ \pi \downarrow & \searrow q & \downarrow r \\ G & \xrightarrow{\quad} & X, \end{array}$$

where $s^{-1}T$ is a closed immersion. By [NS10, Proposition 7.3], $G \rightarrow X$ makes X a coarse moduli space for G .

(iii). That $G \rightarrow X$ is bijective on \mathbb{k} -points follows from [RS07, Theorem 1.2(4)(5)].

(*) . By [NS10, Theorem 5.11], ${}_\ell Z_\infty$ is noetherian. Thus the closed substack Y_∞ is noetherian.

(†). If \mathcal{M} is a coherent graded right \mathcal{R} -module, it follows as in [KRS05, Lemma 5.5] that $\mathcal{M}[1]_n \cong \mathcal{M}_{n+1}^{\sigma^{-1}}$. If \mathcal{M} is a (truncated) point module, therefore, we have $\text{Supp}(\mathcal{M}[1]) = \sigma(\text{Supp}(\mathcal{M}))$. It follows that $q\Psi = \sigma q$. \square

Corollary 3.5. *Let $X, \mathcal{L}, \sigma, P$ be as above and let $R := R(X, \mathcal{L}, \sigma, P)$. Let Y_∞ be the point space of R . If R is generated in degree 1, there is a morphism $p : Y_\infty \rightarrow X$ satisfying the hypotheses of Theorem 1.2.*

Proof. Hypotheses (i), (iii), (iv) follow directly from Proposition 3.4, with $p = q$. Hypothesis (ii) holds by assumption, since the points in P have critically dense, and so dense, orbits. Any point with a dense orbit is contained in the nonsingular locus of X . \square

4. CANONICAL BIRATIONALLY COMMUTATIVE FACTORS AND POINT PARAMETER RINGS

In this section we construct a factor of R universal for maps from R to birationally commutative algebras. Our results hold for any connected graded algebra generated in degree 1 over an algebraically closed field \mathbb{k} , and we work in that generality.

We begin with some more notation for point schemes.

Notation 4.1. Let R be a connected graded \mathbb{k} -algebra generated in degree 1, and adopt Notation 2.4. For $N \in \mathbb{N}$, let

$$Y_N^e := \bigcap_{i \in \mathbb{N}} \phi^i \psi^i(Y_{2i+N}).$$

We refer to Y_N^e as the *essential part* of Y_N . Let $Y_\infty^e := \varprojlim_\phi Y_N^e = \bigcap_k \Psi^k(Y_\infty)$. Let $Y'_n := \bigcap_i \phi^i(Y_{n+i}) = \Phi_n(Y_\infty)$.

Let $\mathbb{P} := \mathbb{P}(R_1^*)$. Then Y_n naturally embeds in $\mathbb{P}^{\times n}$ for any $n \in \mathbb{N}$, and there is a commutative diagram

$$(4.2) \quad \begin{array}{ccc} Y_n & \longrightarrow & \mathbb{P}^{\times n} \\ \phi_n \downarrow & \psi_n & \downarrow \alpha_n \beta_n \\ Y_{n-1} & \longrightarrow & \mathbb{P}^{\times (n-1)}. \end{array}$$

Here $\alpha_n(a_1, \dots, a_n) = (a_1, \dots, a_{n-1})$, and $\beta_n(a_1, \dots, a_n) = (a_2, \dots, a_n)$. Let $\mathcal{M}_n := \mathcal{O}(1, \dots, 1)|_{Y_n}$. Let $\mathcal{M}'_n := \mathcal{M}_n|_{Y'_n}$, and let $\mathcal{M}_n^e := \mathcal{M}_n|_{Y_n^e}$.

We will want to be able to restrict to the essential part Y_∞^e . To do this, we use:

Lemma 4.3. *Let R be any connected graded \mathbb{k} -algebra generated in degree 1, and adopt Notation 4.1. Then*

$$Y_N^e = \bigcap_{i, j \in \mathbb{N}} \phi^i \psi^j(Y_{i+j+N})$$

for any $N \in \mathbb{N}$. Further,

$$\psi(Y_{N+1}^e) = \phi(Y_{N+1}^e) = Y_N^e.$$

Thus, $\Phi_N(Y_\infty^e) = Y_N^e$ (in particular, $Y_N^e \subseteq Y_N'$), and $\Psi(Y_\infty^e) = Y_\infty^e$.

Proof. Fix $N \in \mathbb{N}$. For the first statement, it suffices to show that $\{\phi^k \psi^k(Y_{2k+N})\}_k$ is coinitial in $\{\phi^i \psi^j(Y_{i+j+N})\}_{i,j}$. Let $i, j \in \mathbb{N}$, and let $k := \max(i, j)$. Then

$$\phi^i \psi^j(Y_{i+j+N}) \supseteq \phi^k \psi^k(Y_{2k+N}),$$

as required. The other statements follow immediately. \square

If R is strongly noetherian and generated in degree 1, then the point schemes Y_n stabilize for $n \gg 0$, and it follows that $Y_n = Y_n' = Y_n^e$ for $n \gg 0$. The need to distinguish carefully among the various point spaces is an unpleasant feature of the non-strongly noetherian case.

We follow [RZ08]. Adopt Notation 4.1. Note that Y_n carries a *universal (truncated) point module* $\mathcal{B}_{\leq n} \cong (\phi^n)^* \mathcal{M}_0 \oplus (\phi^{n-1})^* \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n$. If M is the truncated R_A -point module represented by $\delta_n : \text{Spec } A \rightarrow Y_n$, then we have

$$M \cong (\delta_n)^* \mathcal{B}_{\leq n} \cong (\delta_0)^* \mathcal{M}_0 \oplus (\delta_1)^* \mathcal{M}_1 \oplus \cdots \oplus (\delta_n)^* \mathcal{M}_n$$

We define a sheaf \mathcal{B} on Y_∞ by $\mathcal{B} := \bigoplus_{n \geq 0} (\Phi_n)^* \mathcal{M}_n$ (see [LMB00, Chapter 13] and [Ols07] for basics of sheaves on stacks). An honest point module M is determined by its truncations, so given an R_A -point module M represented by $\delta : \text{Spec } A \rightarrow Y_\infty$, we have $M \cong \delta^* \mathcal{B}$. The existence of \mathcal{B} (together with its R -module structure) is equivalent to the statement that Y_∞ is a fine moduli space for point modules, which was observed in [NS10].

As in [RZ08], it is clear from the diagram (4.2) that

$$(4.4) \quad \mathcal{M}_{n+m} \cong (\phi^m)^* \mathcal{M}_n \otimes (\psi^n)^* \mathcal{M}_m,$$

and similarly for \mathcal{M}'_n and \mathcal{M}_n^e . Consider the natural maps $\mathcal{M}_n \rightarrow (\phi^m)_* (\phi^m)^* \mathcal{M}_n$ and $\mathcal{M}_m \rightarrow (\psi^n)_* (\psi^n)^* \mathcal{M}_m$. We obtain maps

$$(4.5) \quad \begin{aligned} H^0(Y_n, \mathcal{M}_n) \otimes H^0(Y_m, \mathcal{M}_m) &\rightarrow H^0(Y_n, (\phi^m)_* (\phi^m)^* \mathcal{M}_n) \otimes H^0(Y_m, (\psi^n)_* (\psi^n)^* \mathcal{M}_m) \\ &= H^0(Y_{n+m}, (\phi^m)^* \mathcal{M}_n) \otimes H^0(Y_{n+m}, (\psi^n)^* \mathcal{M}_m) \rightarrow H^0(Y_{n+m}, \mathcal{M}_{n+m}). \end{aligned}$$

The composition gives a multiplicative structure on

$$B := \bigoplus_{n \geq 0} H^0(Y_n, \mathcal{M}_n)$$

and also on

$$B' := \bigoplus_{n \geq 0} H^0(Y'_n, \mathcal{M}'_n) \quad \text{and} \quad B^e := \bigoplus_{n \geq 0} H^0(Y_n^e, \mathcal{M}_n^e).$$

By an elementary calculation, these multiplications are associative. There are obvious algebra homomorphisms $B \rightarrow B' \rightarrow B^e$.

The ring B is referred to as the *point parameter ring* of R , and its basic properties are worked out in [RZ08]. If V is a vector space, we write $T(V)$ for the (free) tensor algebra on V .

Lemma 4.6. ([RZ08, Lemma 4.1]) *The natural map $T(R_1) \rightarrow B$ factors through R to induce a homomorphism $\theta : R \rightarrow B$. An element a is in $(\ker \theta)_n$ if and only if for all commutative \mathbb{k} -algebras A and all truncated R_A -point modules M of length $n+1$, we have $M_0 a = 0$.*

The map θ induces homomorphisms $\theta' : R \rightarrow B'$ and $\theta^e : R \rightarrow B^e$. We obtain immediately the following universal property of θ' :

Corollary 4.7. *Let R be a connected graded \mathbb{k} -algebra generated in degree 1, and define $\theta' : R \rightarrow B'$ as above. Then*

$$\begin{aligned} \ker \theta' &= \bigcap \{ \text{Ann}_R(M_0) \mid M \text{ is an } R_A\text{-point module for some commutative } \mathbb{k}\text{-algebra } A \} \\ &= \bigcap \{ \text{Ann}_R(M) \mid M \text{ is an } R_A\text{-point module for some commutative } \mathbb{k}\text{-algebra } A \}. \end{aligned}$$

Proof. The first equality follows immediately from Lemma 4.6 and the definition of Y'_n .

For the second, let M be an R_A -point module and let $a \in \ker \theta'$. Then $M_k a = (\Psi^k M)_0 a = 0$ for any $k \in \mathbb{N}$, so $M a = 0$. Thus $\ker \theta' \subseteq \bigcap_M \text{Ann}_R M \subseteq \bigcap_M \text{Ann}_R(M_0) = \ker \theta'$. \square

We are most interested in the ring B^e and the map $\theta^e : R \rightarrow B^e$. Here we have an important universal property, which holds in very large generality: we do not even need R to be noetherian.

We define:

Definition 4.8. A *birationally commutative algebra* is a subalgebra of a skew polynomial extension of a commutative noetherian \mathbb{k} -algebra.

Note that for our purposes here, we will require the commutative algebra in Definition 4.8 to be noetherian, although we caution that this is non-standard.

Then we have:

Theorem 4.9. *Let R be a connected graded \mathbb{k} -algebra, generated in degree 1, and define $\theta^e : R \rightarrow B^e$ as above. Let $\alpha : R \rightarrow \Delta$ be a homomorphism of graded \mathbb{k} -algebras, where Δ is birationally commutative. Then α factors through θ^e up to finite dimension; that is, there is some $n \in \mathbb{N}$ so that $\ker \alpha \supseteq (\ker \theta^e)_{\geq n}$.*

Proof. Without loss of generality, we may assume that $\Delta = A[t; \tau]$, where A is a commutative noetherian \mathbb{k} -algebra and $\tau \in \text{Aut}_{\mathbb{k}}(A)$. Let $S := \alpha(R)$, and let $I := A(S_1 t^{-1}) \subseteq A$. Adopt Notation 2.4.

First suppose that $I = A$. Then the natural map $R_A \rightarrow \Delta$ is surjective, so Δ is an R_A -point module. Since Y_∞ parameterizes point modules, there is a morphism $\delta : \text{Spec } A \rightarrow Y_\infty$, so that $At^n = (\delta_n)^* \mathcal{M}_n = \delta^*(\Phi_n)^* \mathcal{M}_n$ for all $n \in \mathbb{N}$.

Since $\Delta[1]_{\geq 0} \cong \Delta^\tau$ (as an R_A -module), the diagram

$$\begin{array}{ccc} \text{Spec } A & \xrightarrow{\delta} & Y_\infty \\ \tau \downarrow & & \downarrow \Psi \\ \text{Spec } A & \xrightarrow{\delta} & Y_\infty \end{array}$$

commutes. Note that this shows the (scheme-theoretic) image of δ is contained in Y_∞^e , as $\tau \in \text{Aut}_{\mathbb{k}}(A)$. From the maps

$$B^e = \bigoplus H^0(Y_n^e, \mathcal{M}_n^e) \longrightarrow \bigoplus H^0(\text{Spec } A, (\delta_n)^* \mathcal{M}_n^e) = \Delta,$$

we see that α factors through θ^e .

Now suppose that I is a proper ideal of A . Let $0 = J_1 \cap \cdots \cap J_s$ be a minimal primary decomposition of the ideal 0 of A , where J_j is P_j -primary. Reorder the P_j so that P_1, \dots, P_r are the primes containing some $\tau^i(I)$, and P_{r+1}, \dots, P_s are the primes that do not contain any $\tau^i(I)$. Let $K := J_1 \cap \cdots \cap J_r$, and let $K' := J_{r+1} \cap \cdots \cap J_s$; we may have $K = A$ or $K' = A$. However, $K \cap K' = 0$, and K and K' are τ -invariant.

Since τ permutes the finite set P_1, \dots, P_r , there is some N so that $I\tau(I) \cdots \tau^{N-1}(I) \subseteq K$. Thus $S_n t^{-n} \cap K' \subseteq K \cap K' = 0$ for $n \geq N$. If $K' = A$ then this means that S is finite-dimensional, so the result holds. Otherwise, let $\bar{\tau}$ be the induced automorphism of A/K' , and let $\pi : A[t; \tau] \rightarrow (A/K')[t; \bar{\tau}]$ be the natural map. We see that $(\ker \alpha)_n = (\ker \pi \alpha)_n$ for $n \geq N$.

It thus suffices to prove the proposition in the case that I is not contained in any associated prime of A . Assume this is so. Let $A' := Q(A)$, the total ring of quotients of A . By prime avoidance [Eis95, Lemma 3.3], I is not contained in the union of the associated primes of A , and so by [Eis95, Theorem 3.1(b)], I contains a regular element of A . Thus $IA' = A'$. Note that τ extends to an automorphism τ' of A' . There is an induced map $\alpha' : R \rightarrow A'[t; \tau']$; and as $IA' = A'$ this factors through θ^e by the first part of the proof. But $A \subseteq A'$, so $\ker \alpha = \ker \alpha'$ and α also factors through θ^e . \square

We note that in the situation of the last paragraph of the proof, $A'[t; \tau']$ is torsion-free. For, if $0 \neq a \in A'$, then $A'at^k \cdot R_n = aA't^{n+k} \neq 0$, for any n, k . Thus we have shown:

Proposition 4.10. *Let R be a connected graded \mathbb{k} -algebra generated in degree 1. Let $g : R \rightarrow A[t; \tau]$ be a map of graded rings, where A is commutative noetherian and $\tau \in \text{Aut}_{\mathbb{k}}(A)$. Let $I := \ker g$. Then there is a map $\pi : A[t; \tau] \rightarrow A'[t; \tau']$ of commutative graded \mathbb{k} -algebras, where A' is a noetherian total ring of quotients with an automorphism τ' extending τ , so that $\ker \pi g$ is equal to the saturation of I . Further, $\pi(g(R))A' = A'[t; \tau']$. \square*

Theorem 4.9 shows that θ^e is universal for maps from R to birationally commutative algebras. We do not believe this has been observed before. Motivated by this result, we (loosely) refer to the image $\theta^e(R)$ as the *canonical birationally commutative factor* of R ; but note Example 4.11 by way of caution.

Example 4.11. Let V be a $d+1$ -dimensional \mathbb{k} -vector space and let $R := T(V)$ be the free algebra on V . Then $Y_n \cong (\mathbb{P}^d)^{\times n}$, and $B = B' = B^e = \bigoplus_n H^0(Y_n, \mathcal{O}(1, \dots, 1)) \cong R$. Thus $\theta^e(R)$ itself may not be birationally commutative. We thank Chelsea Walton for pointing out this example.

The differences between B^e , B' and B , or alternatively between θ^e , θ' and θ , are fairly subtle. If R is strongly noetherian, then the Y_n stabilize, as mentioned. Then [RZ08, Theorem 1.1] shows that B (and therefore B' and B^e) is equal in large degree to a twisted homogeneous coordinate ring on the projective scheme Y_∞ , and that the map $g : R \rightarrow B$ is surjective in large degree. In fact, we have:

Theorem 4.12. *Let R be a connected graded strongly noetherian algebra generated in degree 1, and let $g : R \rightarrow B(X, \mathcal{L}, \sigma)$ be the map constructed in Theorem 1.1. Then g is universal for maps from R to birationally commutative algebras, and $\ker g$, $\ker \theta^e$, $\ker \theta'$, and $\ker \theta$ are all equal in large degree. Likewise, B , B' , B^e , and $B(X, \mathcal{L}, \sigma)$ are all equal in large degree.*

Proof. We certainly have $\ker \theta \subseteq \ker \theta' \subseteq \ker \theta^e$. Since $B(X, \mathcal{L}, \sigma)$ is birationally commutative, $(\ker \theta^e)_n \subseteq (\ker g)_n$ for $n \gg 0$. By [AZ01, Corollary E4.12], for $n \gg 0$ we have that $\phi_n, \psi_n : Y_n \rightarrow Y_{n-1}$ are isomorphisms. It follows that $Y_n = Y_n^e$ for $n \gg 0$ and that $Y_\infty = Y_\infty^e$ is a projective scheme, isomorphic to Y_n for $n \gg 0$. The construction in the proof of [RZ08, Theorem 1.1] gives $X = Y_\infty$, so $B(X, \mathcal{L}, \sigma)$ and B are equal in large degree. Thus $\ker g$ and $\ker \theta$ are equal in large degree. Since by [RZ08, Theorem 1.1] g is surjective in large degree, B , B' , B^e , and $B(X, \mathcal{L}, \sigma)$ are equal in large degree.

The universal property of θ^e clearly also applies to g . \square

In fact, Theorem 4.12 holds more generally: it is enough to assume that R has subexponential growth and that ϕ_n and ψ_n are isomorphisms for $n \gg 0$. See [RZ08, Theorem 4.4].

We suspect that if R is noetherian, then $\ker \theta$ and $\ker \theta^e$ are equal in large degree; we will see later that, in the situation of Theorem 1.2, $\ker \theta'$ and $\ker \theta^e$ are in fact equal. It is possible that θ^e is always surjective (in large degree); we do not know of an example of a ring generated in degree 1 where this fails. Even under the hypotheses of Theorem 1.2, however, we have not proved these statements.

Our next result shows that if R is noetherian, there is a bijection between kernels of maps from R to birationally commutative algebras (up to finite dimension) and an appropriate class of Ψ -invariant substacks of Y_∞ . This generalizes [RS09, Proposition 3.3].

Definition 4.13. An fpqc-algebraic stack X_∞ is a *Stafford stack* if there is a morphism $\delta : \operatorname{Spec} A \rightarrow X_\infty$, where A is a commutative noetherian \mathbb{k} -algebra, so that X_∞ is the closure of the scheme-theoretic image of δ .

A Stafford stack has finitely many irreducible components and is noetherian at generic points; it may not be noetherian itself.

Proposition 4.14. *Let R be a connected graded noetherian \mathbb{k} -algebra, generated in degree 1, and adopt Notation 2.4. There is a bijection between*

$\{ \sim\text{-equivalence classes of kernels of homomorphisms from } R \text{ to a birationally commutative algebra} \}$

and

$\{ \Psi\text{-invariant closed Stafford substacks } X_\infty \subseteq Y_\infty \}$.

Before proving this, we give a lemma.

Lemma 4.15. *Let $h : \operatorname{Spec} A \rightarrow Y$ be a morphism of schemes, where A is a total ring of quotients of an affine commutative \mathbb{k} -algebra. Let \mathcal{M} be an invertible sheaf on Y . Let $L := H^0(\operatorname{Spec} A, h^* \mathcal{M})$. Then L is a rank 1 free A -module.*

Proof. The maximal ideals of A are the maximal associated primes of A , and there are finitely many of these. Thus A is semilocal, and is a direct sum of finitely many indecomposable semilocal rings, $A \cong A_1 \oplus \cdots \oplus A_k$. Each $L \otimes_A A_i$ is locally free of rank 1 over A_i . By [Hin62], $L \otimes_A A_i$ is actually free over A_i . Thus $L \cong A$. \square

Proof of Proposition 4.14. Let $\alpha : R \rightarrow A[t; \tau]$ be a map of graded \mathbb{k} -algebras, where A is commutative noetherian and $\tau \in \operatorname{Aut}_{\mathbb{k}}(A)$. We are only interested in the \sim -equivalence class of $I := \ker \alpha$, and so we may assume without loss of generality that I is maximal within its \sim -equivalence class. By Proposition 4.10, this means that I is saturated, and we may assume that A is a total ring of quotients and that $A[t; \tau]$ is an R_A -point module. There is thus an induced map $\delta : \operatorname{Spec} A \rightarrow Y_\infty$, with $\Psi \delta = \delta \tau$. Let $X_n := \delta_n(\operatorname{Spec} A) \subseteq Y_n$, and let $X_\infty := \varprojlim X_n$. As τ is an automorphism, X_∞ is Ψ -invariant. It is a Stafford substack of Y_∞ by construction.

This construction defines a map

$$\rho : \{ \text{ideals } I \text{ of } R \text{ so that } R/I \text{ is birationally commutative} \} \rightarrow \{ \text{substacks of } Y_\infty \}$$

that is constant on \sim -equivalence classes. Conversely, let A be commutative noetherian and let $\text{Spec } A \xrightarrow{\delta} X_\infty \subseteq Y_\infty$ be a Ψ -invariant closed Stafford substack of Y_∞ . We must obtain a homomorphism from R to some $A'[t; \tau]$. Note that we may not be able to localize A to obtain A' .

Instead, let $X_n := \Phi_n(X_\infty)$. Each X_n is a closed subscheme of Y_n with $\phi(X_n) = \psi(X_n) = X_{n-1}$. Let G be the finite set of associated primes of A and let $G_n := \delta_n(G)$. Clearly $X_n = \overline{G_n}$. By Proposition 2.5, there is some $N \in \mathbb{N}$ so that both ϕ_n and ψ_n are local isomorphisms at each point of G_n for $n \geq N$. It follows, after possibly increasing N , that both $\phi_n|_{X_n} : X_n \rightarrow X_{n-1}$ and $\psi_n|_{X_n} : X_n \rightarrow X_{n-1}$ are also local isomorphisms at each point of G_n for $n \geq N$.

Let \mathcal{K}_n be the sheaf of total quotient rings of X_n . Then $H^0(X_n, \mathcal{K}_n)$ is a total ring of quotients, and $\phi^*, \psi^* : H^0(X_n, \mathcal{K}_n) \rightarrow H^0(X_{n+1}, \mathcal{K}_{n+1})$ are ring isomorphisms for $n \geq N$. Let $A' := H^0(X_N, \mathcal{K}_N)$.

Note that A' is a localization of an affine algebra, since $X_N \subseteq Y_N$ is of finite type. Let $i_N : \text{Spec } A' \rightarrow X_N$ be the inclusion, and let $i_n := \phi^{N-n} i_N : \text{Spec } A' \rightarrow X_n \subseteq Y_n$ for $n \in \mathbb{N}$. (Our choice of N ensures this is always well-defined.) We thus obtain a map $i : \text{Spec } A' \rightarrow Y_\infty$. Clearly, $\overline{i(\text{Spec } A')} = \overline{\delta(\text{Spec } A)} = X_\infty$.

Let $\tau' := i_N^{-1} \psi_{N+1} \phi_{N+1}^{-1} i_N : \text{Spec } A' \rightarrow \text{Spec } A'$, and let $\tau' : A' \rightarrow A'$ also denote the induced algebra automorphism. We have

$$(4.16) \quad \psi_{n+1} i_{n+1} = i_n \tau'$$

for all $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, let $\mathcal{L}_n := i_n^* \mathcal{M}_n$. Let $S := \bigoplus_{n \geq 0} H^0(\text{Spec } A', \mathcal{L}_n)$. This is an $R_{A'}$ -point module, and also a \mathbb{k} -algebra, with multiplication induced from (4.4). By Lemma 4.15 each S_n is free of rank 1 over A' . From (4.16), we see that $S \cong A'[t; \tau']$. There is a natural algebra map $R \rightarrow B \rightarrow S$.

Let

$$\rho' : \{ \Psi\text{-invariant closed Stafford substacks of } Y_\infty \} \rightarrow \{ \text{ideals of } R \}$$

be the map given by setting $\rho'(X_\infty)$ to be the kernel of the composition $R \rightarrow S \cong A'[t; \tau']$. Since $\overline{i(\text{Spec } A')} = \overline{\delta(\text{Spec } A)}$, therefore $\rho\rho'(X_\infty) = X_\infty$.

On the other hand, suppose that $\text{Spec } A \xrightarrow{\delta} X_\infty \subseteq Y_\infty$ is given by ρ from $\alpha : R \rightarrow A[t; \tau]$, where without loss of generality A is a noetherian total ring of quotients and $A[t; \tau]$ is an R_A -point module. Then applying ρ' gives a total ring of quotients A' , an automorphism τ' of A' , and a dominant map $\text{Spec } A \rightarrow \text{Spec } A'$; that is, an inclusion $A' \subseteq A$. It is clear that $\tau|_{A'} = \tau'$. We thus obtain an inclusion $A'[t; \tau'] \rightarrow A[t; \tau]$, where the diagram

$$\begin{array}{ccc} R & \longrightarrow & A'[t; \tau'] \\ & \searrow \alpha & \downarrow \subseteq \\ & & A[t; \tau] \end{array}$$

commutes. Thus $\rho'\rho(\ker \alpha) = \ker(R \rightarrow A'[t; \tau']) = \ker \alpha$, which is saturated by Proposition 4.10.

Thus ρ and ρ' give inverse bijections, as claimed. \square

From the proof we obtain:

Corollary 4.17. *Let R be a connected graded noetherian \mathbb{k} -algebra, generated in degree 1, and adopt Notation 2.4. Let X_∞ be a Ψ -invariant closed Stafford substack of Y_∞ . Then the saturated ideal of R associated to X_∞ via the correspondence in Proposition 4.14 is the kernel of a homomorphism of graded algebras*

$$g : R \rightarrow A'[t; \sigma],$$

where A' is the total ring of quotients constructed in the proof of Proposition 4.14, and $\sigma \in \text{Aut}(A')$. In particular, any Ψ -invariant closed Stafford substack of Y_∞ is induced from a map from a scheme essentially of finite type to Y_∞ . \square

We refer to the algebra A' above as the *total ring of quotients* of X_∞ .

Remark 4.18. The bijection defined in Proposition 4.14 is easily seen to be order-reversing in the following sense: if $Z_\infty \subseteq X_\infty$ are Ψ -invariant closed Stafford substacks of Y_∞ , then the corresponding ideals satisfy $(I_Z)_{\geq n} \supseteq (I_X)_{\geq n}$ for $n \gg 0$.

5. COARSE MODULI: FIRST PROPERTIES

We now begin to work towards the proof of Theorem 1.2. In this section, we assume the existence of a coarse moduli space $G \rightarrow X$. We construct an induced automorphism σ of X and study some of its properties.

Let R be a connected graded noetherian \mathbb{k} -algebra generated in degree 1, and adopt Notation 2.4. We will say that the projective scheme X is a *coarse moduli scheme for G* if there is a commutative diagram

$$\begin{array}{ccc} & F \cong Y_\infty & \\ \pi \swarrow & & \searrow p \\ G & \xrightarrow{\quad} & X, \end{array}$$

so that the map $G \rightarrow X$ makes X a coarse moduli space for G . We will say that p is an *isomorphism in codimension d* if p^{-1} is defined at all points in X of codimension d .

Proposition 5.1. *Let R be a connected graded noetherian \mathbb{k} -algebra generated in degree 1, and adopt Notation 4.1. Further suppose that the projective scheme X is a coarse moduli scheme for G . Then:*

- (1) *The endomorphism Ψ of Y_∞ induces an automorphism σ of X ;*
- (2) *$p : Y_\infty^e \rightarrow X$ is scheme-theoretically surjective;*
- (3) *if p is an isomorphism in codimension 0, then there is a \mathbb{k} -algebra homomorphism $g : R \rightarrow K[t; \sigma]$, where K is the ring of (global) fractions on X .*

Proof. (1) We begin with the following observation:

Remark 5.2. The quotient functor $G = F/\sim$ is the sheafification of the colimit of the following directed system:

$$\{Y_\infty, \Psi\} : Y_\infty \xrightarrow{\Psi} Y_\infty \xrightarrow{\Psi} Y_\infty \xrightarrow{\Psi} \dots$$

More precisely, one writes $Y_{\infty, \ell}$ for the copy of Y_∞ indexed by $\ell \geq 0$. Defining a map $Y_{\infty, \ell} \rightarrow G$ that takes $M \mapsto \pi(M[-\ell])$, the image of $M[-\ell]$ in $\text{qgr-}R$, then one evidently realizes G as (the sheafification of) the colimit of the directed system.

We can define an endomorphism of this directed system by:

$$(5.3) \quad \begin{array}{ccccccc} Y_\infty & \xrightarrow{\Psi} & Y_\infty & \xrightarrow{\Psi} & Y_\infty & \xrightarrow{\Psi} & \dots \\ \downarrow \Psi & & \downarrow \Psi & & \downarrow \Psi & & \\ Y_\infty & \xrightarrow{\Psi} & Y_\infty & \xrightarrow{\Psi} & Y_\infty & \xrightarrow{\Psi} & \dots \end{array}$$

This induces a morphism $\Psi : G \rightarrow G$ of sheaffied colimits, and a commutative diagram

$$\begin{array}{ccccc} Y_\infty & \xrightarrow{\Psi} & Y_\infty & & \\ \pi \downarrow & & \downarrow \pi & \searrow p & \\ G & \xrightarrow{\Psi} & G & \xrightarrow{\quad} & X. \end{array}$$

Because X is a coarse moduli space for G , any map $Y_\infty \rightarrow X$ that factors through π must factor through p . Therefore, there is an induced morphism $\sigma : X \rightarrow X$ that yields a commutative diagram:

$$(5.4) \quad \begin{array}{ccccc} Y_\infty & \xrightarrow{\Psi} & Y_\infty & & \\ \pi \downarrow & \searrow p & & \searrow p & \\ G & \xrightarrow{\quad} & X & \xrightarrow{\sigma} & X. \end{array}$$

Besides the “obvious” identity self-map of the directed system $\{Y_\infty, \Psi\}$, there is also the self-map

$$(5.5) \quad \begin{array}{ccccccc} Y_\infty & \xrightarrow{\Psi} & Y_\infty & \xrightarrow{\Psi} & Y_\infty & \xrightarrow{\Psi} & \dots \\ & \searrow \Psi & & \searrow \Psi & & \searrow \Psi & \\ Y_\infty & \xrightarrow{\Psi} & Y_\infty & \xrightarrow{\Psi} & Y_\infty & \xrightarrow{\Psi} & \dots \end{array}$$

From the description of G given by Remark 5.2, this self-map induces the identity functor $\text{id} : G \rightarrow G$ on sheafified colimits and hence the identity automorphism of X as well. Finally, observe that the composite of (5.3) and

$$(5.6) \quad \begin{array}{ccccccc} Y_\infty & \xrightarrow{\Psi} & Y_\infty & \xrightarrow{\Psi} & Y_\infty & \xrightarrow{\Psi} & \dots \\ & \searrow \text{id} & & \searrow \text{id} & & \searrow \text{id} & \\ Y_\infty & \xrightarrow{\Psi} & Y_\infty & \xrightarrow{\Psi} & Y_\infty & \xrightarrow{\Psi} & \dots \end{array}$$

equals (5.5). Hence (5.6) induces an inverse to σ on X , and in particular σ is an automorphism.

(2) It follows immediately from commutativity of (5.4) that $X = \bigcap_n \sigma^n(X) = \bigcap_n p\Psi^n(Y_\infty) = p(Y_\infty^e)$.

(3) Since p is an isomorphism in codimension 0, p^{-1} is defined at the generic point of all components of X . We immediately obtain a Ψ -invariant Stafford substack X_∞ of Y_∞ whose total ring of quotients is K . Then the result follows from Corollary 4.17. \square

In this generality, we cannot say much about the map g or about $g(R)$ —we do not know, for example, if $g(R)$ is the canonical birationally commutative factor of R up to finite dimension. We do note that if we assume that p^{-1} is defined in codimension 1 and X is locally factorial at points in the indeterminacy locus of p^{-1} , then it is not hard to show (using an argument similar to that in Proposition 5.9) that $g(R)$ is contained in a twisted homogeneous coordinate ring $B(X, \mathcal{L}, \sigma)$, where \mathcal{L} is an invertible sheaf on X . We do not know, however, if \mathcal{L} must be σ -ample.

We note that Proposition 5.1 applies to the situation in Theorem 1.2 that we are most interested in.

Lemma 5.7. *Adopt Notation 2.4, and assume that the hypotheses of Theorem 1.2 hold. Then the morphism p is birational. More precisely, let η be the generic point of X , with function field $K = \mathbb{k}(\eta)$. Then p^{-1} is defined and is a local isomorphism at η . Thus conclusions (1)–(3) of Proposition 5.1 hold for R .*

Proof. Adopt Notation 4.1. By Proposition 2.1, $p : Y_\infty \rightarrow X$ factors through Φ_{n_0} for some n_0 . Let $f_{n_0} : Y'_{n_0} \rightarrow X$ be the induced map. For $n \geq n_0$, let

$$f_n := f_{n_0} \phi^{n-n_0} : Y'_n \rightarrow X.$$

Let Ω be the indeterminacy locus of p^{-1} ; by assumption, Ω consists of countably many \mathbb{k} -points. If $n \geq n_0$, then the indeterminacy locus of f_n^{-1} is a closed subset of X that is contained in Ω : it is thus finite. Thus f_n^{-1} is defined at η ; taking the limit, we see that p^{-1} is defined at η . \square

We now suppose that the coarse moduli space X exists, and in addition that $p : Y_\infty \rightarrow X$ is an isomorphism in codimension 1. This allows us to construct a sequence of reflexive sheaves on X , whose properties we now study.

Notation 5.8. Adopt Notation 4.1, and suppose that there is a projective variety X that is a coarse moduli scheme for G . Suppose in addition that p is an isomorphism in codimension 1.

Using Proposition 2.1, let n_0 be such that p factors through Φ_n for $n \geq n_0$, and let $f_n : Y'_n \rightarrow X$ be the induced map. Note that $(f_n)^{-1}$ is defined in codimension 1 for $n \geq n_0$.

For $n \geq n_0$, let W_n be the indeterminacy locus of $(f_n)^{-1} : X \dashrightarrow Y'_n$. By assumption, W_n has codimension 2 in X . Let $U_n := X \setminus W_n$ and let $i_n : U_n \rightarrow X$ be the inclusion morphism.

For $n \geq n_0$, let $\mathcal{R}'_n := (f_n)_* \mathcal{M}'_n$ and let $\mathcal{N}_n := (i_n)_* (i_n)^* \mathcal{R}'_n$. For $1 \leq n < n_0$, let $\mathcal{R}'_n := (f_{n_0})_* (\phi^{n_0-n})^* \mathcal{M}'_n$ and let $\mathcal{N}_n := (i_{n_0})_* (i_{n_0})^* \mathcal{R}'_n$.

Let $K := \mathbb{k}(X)$. Let $\sigma \in \text{Aut}(X)$ and $g : R \rightarrow K[t; \sigma]$ be given by Proposition 5.1. Let $\mathcal{L} := (\mathcal{N}_{n_0+1} \otimes (\mathcal{N}_{n_0}^\vee)^\sigma)^{\vee\vee}$. For $n \geq 0$, let

$$\mathcal{L}_n := (\mathcal{L} \otimes \mathcal{L}^\sigma \otimes \dots \otimes \mathcal{L}^{\sigma^{n-1}})^{\vee\vee}.$$

Let $\mathcal{R}'_0 := \mathcal{N}'_0 := \mathcal{O}_X$.

We collect some basic properties of the sheaves \mathcal{N}_n , \mathcal{L}_n , and \mathcal{R}'_n .

Proposition 5.9. *Adopt Notation 5.8. In particular, suppose that there is a projective variety X that is a coarse moduli scheme for G and that p is an isomorphism in codimension 1.*

- (1) *The sheaves \mathcal{N}_n are reflexive for $n \geq 0$, and $\mathcal{N}_n = (\mathcal{R}'_n)^{\vee\vee}$.*
- (2) *For $n \geq 0$ the natural map $\mathcal{R}'_n \rightarrow \mathcal{N}_n$ is an isomorphism in codimension 1.*

(3) There are “multiplication” maps

$$\mathcal{R}'_n \otimes (\mathcal{R}'_m)^{\sigma^n} \rightarrow \mathcal{R}'_{n+m}$$

for all $n \geq 0$ and $m \geq n_0$, satisfying the obvious associativity conditions. In particular, $\mathcal{O}_X \oplus \bigoplus_{n \geq n_0} \mathcal{R}_n$ is a bimodule algebra. (See Section 3.)

(4) For any $k \geq n_0$, we have $\mathcal{L} \cong (\mathcal{N}_{k+1} \otimes (\mathcal{N}_k^\vee)^\sigma)^{\vee\vee}$.

(5) In fact, for any $n \geq 0$, we have $\mathcal{N}_n \cong \mathcal{L}_n$.

Proof. (1), (2). Let $n \geq n_0$ and let $i := i_n$. Away from the codimension 2 set W_n , the sheaf \mathcal{R}'_n is invertible, as f_n is a local isomorphism. The kernel and cokernel of $\mathcal{R}'_n \rightarrow \mathcal{N}_n$ are supported on W_n . This proves (1) and (2) for $n \geq n_0$; the proof for $n < n_0$ is similar, using W_{n_0} .

(3). Let $m, n \geq n_0$. The natural morphisms

$$\mathcal{M}'_n \rightarrow (\phi^m)_*(\phi^m)^*\mathcal{M}'_n$$

and

$$\mathcal{M}'_m \rightarrow (\psi^n)_*(\psi^n)^*\mathcal{M}'_m$$

induce maps

$$\mathcal{R}'_n \rightarrow (f_n)_*(\phi^m)_*(\phi^m)^*\mathcal{M}'_n = (f_{n+m})_*(\phi^m)^*\mathcal{M}'_n$$

and

$$(\sigma^n)^*\mathcal{R}'_m \rightarrow (\sigma^n)^*(f_m)_*(\psi^n)_*(\psi^n)^*\mathcal{M}'_m = (\sigma^n)^*(\sigma^n)_*(f_{m+n})_*(\psi^n)^*\mathcal{M}'_m = (f_{m+n})_*(\psi^n)^*\mathcal{M}'_m.$$

If $0 \leq n < n_0$ and $m \geq n_0$, there is a map

$$\mathcal{R}'_n = (f_{n_0})_*(\phi^{n_0-n})^*\mathcal{M}'_n \rightarrow (f_{n_0})_*(\phi^{m-n_0+n})^*(\phi^{m-n_0+n})^*(\phi^{n_0-n})^*\mathcal{M}'_n = (f_{n+m})_*(\phi^m)^*\mathcal{M}'_n.$$

For all $n \geq 0$ and $m \geq n_0$, the isomorphism

$$(5.10) \quad (\phi^m)^*\mathcal{M}'_n \otimes (\psi^n)^*\mathcal{M}'_m \rightarrow \mathcal{M}'_{n+m}$$

observed in Section 4 thus induces a multiplication map

$$\mathcal{R}'_n \otimes (\mathcal{R}'_m)^{\sigma^n} \rightarrow (f_{n+m})_*(\phi^m)^*\mathcal{M}'_n \otimes (f_{n+m})_*(\psi^n)^*\mathcal{M}'_m \rightarrow (f_{n+m})_*\mathcal{M}'_{n+m} = \mathcal{R}'_{n+m}.$$

(4). If $k > n_0$, then from (5.10) we have

$$\phi^*(\mathcal{M}'_k \otimes (\psi^*\mathcal{M}'_{k-1})^{-1}) \cong \mathcal{O}(1, 0^k)|_{Y'_{k+1}} \cong \mathcal{M}'_{k+1} \otimes (\psi^*\mathcal{M}'_k)^{-1}.$$

Working on the open set where f_{k+1} is an isomorphism, we see that the isomorphism class of \mathcal{L} does not depend on the integer $k \geq n_0$ used to define it.

(5). The multiplication maps induce morphisms $\mathcal{N}_n \otimes \mathcal{N}_m^{\sigma^n} \rightarrow \mathcal{N}_{n+m}$ for all $n \geq 0, m \geq n_0$. Since f_{n+m} is an isomorphism away from a codimension 2 locus in X and the \mathcal{N}_k are reflexive, the induced maps $(\mathcal{N}_n \otimes \mathcal{N}_m^{\sigma^n})^{\vee\vee} = (i_{n+m})^*i_{n+m}^*(\mathcal{N}_n \otimes \mathcal{N}_m^{\sigma^n}) \rightarrow \mathcal{N}_{n+m}$ must in fact be isomorphisms. Further, these maps are associative, since the corresponding property holds for (5.10).

Let $k \geq n_0$ and $n \geq 0$; then we have

$$\begin{aligned} \mathcal{N}_n &\cong (\mathcal{N}_{n+k} \otimes (\mathcal{N}_k^\vee)^{\sigma^n})^{\vee\vee} \cong \\ &(\mathcal{N}_{n+k} \otimes (\mathcal{N}_{n+k-1}^\vee)^\sigma \otimes \mathcal{N}_{n+k-1}^\sigma \otimes (\mathcal{N}_{n+k-2}^\vee)^{\sigma^2} \otimes \cdots \otimes \mathcal{N}_{k+1}^{\sigma^{n-1}} \otimes (\mathcal{N}_k^\vee)^{\sigma^n})^{\vee\vee} \cong (\mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}})^{\vee\vee} = \mathcal{L}_n. \end{aligned}$$

□

The bimodule $\bigoplus_{n \in \mathbb{N}} (\mathcal{L}_n)_{\sigma^n}$ is also a bimodule algebra. The global sections of a bimodule algebra have a natural algebra structure, and one can show that the natural map $R \rightarrow \bigoplus H^0(X, \mathcal{N}_n) \cong \bigoplus H^0(X, \mathcal{L}_n)$ is in fact an algebra homomorphism. However, it is easier to work instead with the map $g : R \rightarrow K[t; \sigma]$ defined in Notation 5.8 and given by Proposition 5.1, and we do so.

6. POINTS AND CURVES IN Y_∞

To prove Theorem 1.2, we must understand the geometry of point spaces at a finer level, and in particular, study the structure of the countable subset of X consisting of indeterminacy points of p^{-1} . We next focus on curves. Note the next result holds for any connected graded noetherian R generated in degree 1, without further assumptions on the structure of Y_∞ or on the cardinality of \mathbb{k} .

Proposition 6.1. *Let \mathbb{k} be an algebraically closed field, and let R be a connected graded noetherian \mathbb{k} -algebra generated in degree 1. Adopt Notation 2.4. Let C be an irreducible projective curve in Y_n , and suppose that Φ_n^{-1} is defined at the generic point of C . Then C contains only finitely many points of indeterminacy of Φ_n^{-1} .*

Proof. Let \tilde{C} be the normalization of C . Then the map $\tilde{C} \rightarrow C \rightarrow Y_n$ lifts via the morphisms ϕ^m to maps $\tilde{C} \rightarrow Y_{n+m}$ for all $m \geq 0$. These maps stabilize for $m \gg 0$, by finiteness of the integral closure, to induce morphisms

$$\tilde{C} \xrightarrow{j} \hat{C} \xrightarrow{i} Y_\infty$$

for some projective curve \hat{C} , where j is birational and i is a closed immersion.

Let $U = \text{Spec } A$ be an open affine subset of \hat{C} . Let M be the A -point module associated to

$$U \rightarrow \hat{C} \rightarrow Y_\infty.$$

Since A is an affine commutative \mathbb{k} -algebra, the ring R_A is noetherian. Thus the right ideal I given by

$$0 \rightarrow I \rightarrow R_A \rightarrow M \rightarrow 0$$

is finitely generated, in degrees $\leq N$. If x is any \mathbb{k} -point of U , the associated \mathbb{k} -point module is $M_x := M \otimes_A \mathbb{k}_x$. Since M is A -flat, the sequence

$$0 \rightarrow I \otimes_A \mathbb{k}_x \rightarrow R \rightarrow M_x \rightarrow 0$$

remains exact. In particular, the right ideal defining M_x is generated in degrees $\leq N$. This precisely says that for $m \geq N$, the map $\Phi_m : Y_\infty \rightarrow Y_m$ is a local isomorphism at x , and since $x \in U$ was arbitrary, at all points of U .

Covering \hat{C} by finitely many open affines and increasing N if necessary, we see that Φ_N is a local isomorphism at all points of \hat{C} . Thus the only components of the indeterminacy locus of Φ_n^{-1} that C may possibly meet are those in the indeterminacy locus of $(\phi^{n-N})^{-1} : Y_N \dashrightarrow Y_n$. This is a proper closed subscheme of C and thus is finite. \square

In the next few results, we consider the following slightly weaker version of the hypotheses of Theorem 1.2:

Hypotheses 6.2. *Let \mathbb{k} be an uncountable algebraically closed field, and let R be a noetherian connected graded \mathbb{k} -algebra generated in degree 1. Adopt Notation 2.4. Suppose the following:*

- (i) *there is a commutative diagram*

$$\begin{array}{ccc} & F \cong Y_\infty & \\ \pi \swarrow & & \searrow p \\ G & \xrightarrow{\quad} & X \end{array}$$

where X is a projective scheme and a coarse moduli space for G ;

- (ii) *X is a variety of dimension ≥ 2 ;*
 (iii) *the map $G \rightarrow X$ is bijective on \mathbb{k} -points;*
 (iv) *the indeterminacy locus of p^{-1} consists (set-theoretically) of countably many points.*

In particular, we alert the reader that from here on, we will assume that \mathbb{k} is uncountable.

Corollary 6.3. *If Hypotheses 6.2 hold, then any curve in X contains only finitely many points of indeterminacy of p^{-1} .* \square

We next study preimages of \mathbb{k} -points in Y_∞ . We write a (not necessarily closed) point of Y_∞ as $y = (y_n)$, with $y_n = \Phi_n(y) \in Y'_n$.

Proposition 6.4. *Adopt Notation 2.4, and assume that Hypotheses 6.2 hold. Let x be a \mathbb{k} -point of X , and let y, z be (not necessarily closed) points of $p^{-1}(x) \subseteq Y_\infty$. Then $\Psi^k(y) = \Psi^k(z)$ for $k \gg 0$. In particular, $\Psi^k(z)$ is a \mathbb{k} -point of Y_∞ for $k \gg 0$.*

Proof. If y, z are \mathbb{k} -points, this follows from the fact that $G \rightarrow X$ is a bijection on \mathbb{k} -points. It suffices, then, to prove the proposition in the case that z is not a \mathbb{k} -point and y is a \mathbb{k} -point; in fact, by the first sentence of the proof it suffices to prove the proposition for z and for *one* \mathbb{k} -rational point $y' \in p^{-1}(x)$.

Let $z = (z_n)$ and let $Z_n := \overline{\{z_n\}} \subseteq Y'_n$. By Proposition 2.5, there is some $N \in \mathbb{N}$ so that for $n \geq N$ the rational map ϕ_{n+1}^{-1} is defined and is a local isomorphism at z_n . Thus the fiber $\phi_{n+1}^{-1}(z_n)$ is a singleton, equal to z_{n+1} . That is, for $n \geq N$ the morphism $\phi_{n+1} : Z_{n+1} \rightarrow Z_n$ is birational and scheme-theoretically surjective, and $\mathbb{k}(z_n) = \mathbb{k}(z)$. Let $Z_\infty := \varprojlim_\phi Z_n$. Let $y' = (y'_n)$ be a \mathbb{k} -point of $Z_\infty \subseteq Y_\infty$.

For all $k \in \mathbb{N}$, let

$$Z^{(k)} := Z_N \cap \Phi_N((\Psi^k)^{-1}\Psi^k(y')).$$

More formally,

$$Z^{(k)} = Z_N \cap \bigcap_{n \geq N+k} \phi^{n-N}((\psi^k)^{-1}\psi^k(y'_n)).$$

This is a closed subscheme of Z_N , since ϕ is proper and y'_n is a closed point. We have $Z^{(k)} \subseteq Z^{(k+1)}$.

If $y'' \in Z_\infty$ is a \mathbb{k} -point, then $\Psi^k(y'') = \Psi^k(y')$ for some k by the first sentence of the proof. That is, the countable union $\bigcup_k Z^{(k)}$ contains all \mathbb{k} -points of $\Phi_N(Z_\infty) = Z_N$. Since \mathbb{k} is uncountable, we must have some $Z^{(k)} = Z_N$ by the following.

Lemma 6.5. *Suppose Z is an irreducible variety over a field \mathbb{k} of cardinality \aleph . If S is a set of cardinality \beth strictly less than \aleph and $V \subset Z$ is a union of proper closed subsets V_i , $i \in S$, then $Z \setminus V$ contains a closed point.*

We are grateful to Ravi Vakil for the following argument.

Proof. We may assume that each V_i is an irreducible hypersurface and Z is affine. Suppose the statement is true for $Z = \mathbb{A}^n$, where $n = \dim(Z)$. Then Noether normalization gives a finite surjective map $\phi : Z \rightarrow \mathbb{A}^n$ and the images of the V_i under ϕ are irreducible hypersurfaces; if $x \in \mathbb{A}^n \setminus \bigcup_{i \in S} \phi(V_i)$ is closed, so is $\phi^{-1}(x)$. So it suffices to establish the case $Z = \mathbb{A}^n$, which we do by induction; $n = 1$ is clear. For the inductive step, project $\mathbb{A}^n \rightarrow \mathbb{A}^1$; only \beth -many of the fibers can be among the V_i , so we can choose a fiber which is not among the V_i and, by the inductive hypothesis, a closed point in it. \square

Let $n \geq N + k$. As $\phi^{n-N}((\psi^k)^{-1}\psi^k(y'_n)) \ni z_N$, by choice of N we have

$$z_n = (\phi^{n-N})^{-1}(z_N) \in (\psi^k)^{-1}\psi^k(y'_n),$$

and $\psi^k(z_n) = \psi^k(y'_n)$. Thus $\Psi^k(z) = \Psi^k(y')$. \square

Corollary 6.6. *Adopt Notation 2.4, and assume that Hypotheses 6.2 hold. Let x be a σ -fixed \mathbb{k} -point of X . Then $p^{-1}(x) = \{v\}$ is a single \mathbb{k} -point, and Ψ^{-1} is defined and is a local isomorphism at v .*

Proof. Let $z \in p^{-1}(x)$. As $p\Psi(z) = \sigma p(z) = x$, by Proposition 6.4 there is some k so that $\Psi^k(z) = \Psi^{k+1}(z)$ is a \mathbb{k} -point of Y_∞ . Let $v = (v_n) := \Psi^k(z)$. This is a \mathbb{k} -point of $p^{-1}(x)$. As $\Psi(v) = v$, we have $\psi_{n+1}(v_{n+1}) = \phi_{n+1}(v_{n+1}) = v_n$ for all $n \in \mathbb{N}$. By Proposition 2.5, for $n \gg 0$ both ϕ_n^{-1} and ψ_n^{-1} are defined and are local isomorphisms at v_{n-1} . Since $\Psi = \varprojlim_{\phi_n} \psi_n$, it follows that Ψ^{-1} is defined and is a local isomorphism at v . By Proposition 6.4, $p^{-1}(x) = \bigcup_{k \in \mathbb{N}} \Psi^{-k}(v)$. Thus $p^{-1}(x) = \{v\}$ is a single \mathbb{k} -point. \square

We now prove a key result: that the point space Y_∞ differs from the coarse moduli space X only at points of infinite order.

Theorem 6.7. *Adopt Notation 2.4, and assume that Hypotheses 6.2 hold. Then all points where $p^{-1} : X \dashrightarrow Y_\infty$ is undefined have infinite σ -orbits.*

Proof. We adopt Notation 5.8 to obtain maps $f_n : Y'_n \rightarrow X$ for all $n \geq n_0$, where the indeterminacy locus of f_n^{-1} consists of isolated points.

Suppose that $x \in X$ has a finite orbit. By passing to a Veronese of R , and thus replacing σ by σ^n , we may assume that $\sigma(x) = x$. By Corollary 6.6, $p^{-1}(x) = \{v\}$ is a single \mathbb{k} -point. Using Proposition 2.5, let $N \geq n_0$ be such that Φ_N is a local isomorphism at v , and let $v_N := \Phi_N(v)$. Then $f_N^{-1}(x) = \{v_N\}$. We will show that f_N is a local isomorphism at v_N : that is, that f_N^{-1} is defined at x .

Let W_N be the (finite) indeterminacy locus of f_N^{-1} . Let $U_1 := Y'_N \setminus f_N^{-1}(W_N \setminus \{x\})$. Let $U_2 := X \setminus \{x\}$. Let $U_{12} := U_1 \setminus \{v_N\}$ and let $U_{21} := X \setminus W_N$. Let \tilde{X} be the glueing of U_1 and U_2 along the isomorphism $f_N : U_{12} \rightarrow U_{21}$.

For $j = 1, 2$ let $i_j : U_j \rightarrow \tilde{X}$ be the canonical map. Since $i_1 = i_2 f_N$ as a map from U_{12} to X , we may glue i_1 and i_2 to obtain a morphism $i : \tilde{X} \rightarrow X$ which is an isomorphism away from $i_1(v_N) \in \tilde{X}$.

Let $U_3 := Y'_N \setminus \{v_N\}$. Then $U_3 \cup U_1 = Y'_N$ and $U_3 \cap U_1 = U_{12}$. As

$$U_3 \xrightarrow{f_N} U_2 \xrightarrow{i_2} \tilde{X} \quad \text{and} \quad U_1 \xrightarrow{i_1} \tilde{X}$$

agree on U_{12} , we may glue these maps to obtain a morphism $h : Y'_N \rightarrow \tilde{X}$, with $ih = f_N$. It follows from Proposition 5.1(2) that $h|_{U_3}$ is scheme-theoretically surjective as a map from $U_3 \rightarrow U_2$, and so h is scheme-theoretically surjective.

Let A be a commutative local \mathbb{k} -algebra and let $C := \operatorname{Spec} A$. Suppose that $a, b : C \rightarrow Y_\infty$ are morphisms and that $\Psi^k a = \Psi^k b$. We claim that $h\Phi_N a = h\Phi_N b$. It follows that the map $h\Phi_N : Y_\infty \rightarrow \tilde{X}$ factors through $\pi : F \rightarrow G$.

To prove the claim, let \mathfrak{m} be the maximal ideal of A . If $a(\mathfrak{m}) \neq v$ then necessarily $b(\mathfrak{m}) \neq v$ and so $h\Phi_N(\mathfrak{m}) \in i_2(U_2)$. Note that $\sigma(U_2) = U_2$. Then

$$h\Phi_N a = i_2 p a = i_2 \sigma^{-k} p \Psi^k a = i_2 \sigma^{-k} p \Psi^k b = i_2 p b = h\Phi_N b$$

as claimed. On the other hand, suppose that $a(\mathfrak{m}) = v = b(\mathfrak{m})$. As Ψ is a local isomorphism at v we must have $a = b$, so certainly $h\Phi_N a = h\Phi_N b$.

By the universal property of X there is thus a morphism $j : X \rightarrow \tilde{X}$ so that

$$\begin{array}{ccc} Y_\infty & & \\ p \downarrow & \searrow h\Phi_N & \\ X & \xrightarrow{j} & \tilde{X} \end{array}$$

commutes. Clearly $ij = \operatorname{Id}_X$, as ij is the identity on closed points and X is a variety. Thus $jij = j$. As $h\Phi_N = jp$ is scheme-theoretically surjective, j is also scheme-theoretically surjective. By Lemma 2.2 we have $ji = \operatorname{Id}_{\tilde{X}}$. Thus $i = j^{-1}$ is an isomorphism, and both $f_N^{-1} = (ih)^{-1}$ and $p^{-1} = (f_N\Phi_N)^{-1}$ are defined at x . \square

To end the section, we give an important ring-theoretic consequence of Theorem 6.7.

Theorem 6.8. *Let R be a connected graded noetherian algebra generated in degree 1 over an uncountable algebraically closed field \mathbb{k} , and adopt Notation 2.4. Assume that Hypotheses 6.2 hold. Let K be the field of fractions of X . Then there are an automorphism σ of K and a \mathbb{k} -algebra homomorphism $g : R \rightarrow K[t; \sigma]$ so that $\ker g$, $\ker \theta^e$, and $\ker \theta'$ are all equal.*

Proof. The automorphism σ and the map g are given by Proposition 5.1.

We first show that $\ker g$, $\ker \theta'$, and $\ker \theta^e$ are all equal in large degree. By Theorem 4.9 we have $(\ker g)_{\gg 0} \supseteq (\ker \theta^e)_{\gg 0}$. As $\ker \theta^e \supseteq \ker \theta'$, it suffices to prove that $\ker \theta'$ and $\ker g$ are equal in large degree. Let $\overline{R} := \theta'(R)$ and let $J := (\ker g + \ker \theta')/(\ker \theta') \subseteq \overline{R}$. For $n \geq n_0$, let X_n be the strict transform of X in Y'_n . Then

$$J_n = \{h \in \overline{R}_n \subseteq H^0(Y'_n, \mathcal{M}'_n) \mid h|_{X_n} \equiv 0\}.$$

Since R is noetherian, J is finitely generated as a right and a left ideal. Thus for some $N \geq n_0$ there are $b_1, \dots, b_k \in J_N$ that generate $J_{\geq N}$ as both a right and a left ideal.

Let $n \geq N$ and let $h \in J_n$, so $h|_{X_n}$ is identically 0. Then there are $a_1, \dots, a_k, c_1, \dots, c_k \in \overline{R}_{n-N}$ so that

$$h = \sum_i (\phi^{n-N})^* b_i (\psi^N)^* c_i = \sum_i (\phi^N)^* a_i (\psi^{n-N})^* b_i.$$

Now, if $y \in Y'_n$ with $f_n(y) \notin W_N$, then f_N is a local isomorphism at $\phi^{n-N}(y)$. Thus each b_i vanishes in a neighborhood of $\phi^{n-N}(y)$, and so $(\phi^{n-N})^* b_i$ vanishes identically along $(\phi^{n-N})^{-1} \phi^{n-N}(y) = f_n^{-1} f_n(y)$. That is, from the first equation we conclude that h vanishes at all points in $Y'_n \setminus f_n^{-1}(W_N)$. Likewise, it follows from the second equation that h vanishes away from $f_n^{-1}(\sigma^{-(n-N)}(W_N))$. Since by Theorem 6.7 all points in W_N have infinite σ -orbits, for $n \gg 0$ the set $W_N \cap \sigma^{-(n-N)}(W_N) = \emptyset$, and all $h \in J_n$ must vanish identically on Y'_n ; that is, $J_n = 0$ for $n \gg 0$.

We now claim that $\ker g = \ker \theta'$. Let A be a commutative \mathbb{k} -algebra, and let $M = R_A/I$ be an R_A -point module. By Corollary 4.7, $I \supseteq \ker \theta'$, and by the above $\ker \theta'$ and $\ker g$ are equal in large degree. Thus there is some $N \in \mathbb{N}$ so that $I \supseteq (\ker g)_{\geq N}$. Thus $(I + \ker g)/I$ is a torsion submodule of M_R . But we have:

Sublemma 6.9. *Let R be a noetherian \mathbb{k} -algebra generated in degree 1, let A be a commutative \mathbb{k} -algebra, and let M be an R_A -point module. Then M_R is torsion-free.*

Proof. Let $m \in M_i$, and suppose that $mR_k = 0$. By shifting, we may suppose that $i = 0$, and as M is an R_A -point module we may identify M_0 with A and let $m \in A$. By abuse of notation, we let $1 \in M_0$ be the generator of M . Then we have $0 = mR_k A = 1 \cdot mR_k A = 1 \cdot R_k A m = M_k m$. As $M_k \cong A$, we have $m = 0$. \square

Returning to the proof of Theorem 6.8, the torsion submodule $(I + \ker g)/I$ of M_R is 0, so $\ker g \subseteq I$. As $I = \text{Ann}_{R_A}(M_0)$ and M was arbitrary, we have

$$\ker g \subseteq \bigcap \{ \text{Ann}_R(M_0) \mid M \text{ is an } R_A\text{-point module for some commutative } \mathbb{k}\text{-algebra } A \}.$$

This is $\ker \theta'$ by Corollary 4.7. Thus we have $\ker \theta' \subseteq \ker g \subseteq \ker \theta'$ and the two are equal.

We thus have $\ker g = \ker \theta' \subseteq \ker \theta^e$, and the three are equal in large degree. But $g(R) \subseteq K[t; \sigma]$, and this is a domain; hence the torsion ideal $(\ker \theta^e)/(\ker g)$ of $g(R)$ must be 0, and $\ker \theta^e = \ker g = \ker \theta'$. \square

Corollary 6.10. *Assume Hypotheses 6.2 hold, and let $g : R \rightarrow K[t; \sigma]$ be the algebra homomorphism constructed in Proposition 5.1. Then g is universal for maps from R to birationally commutative algebras. In particular, the canonical birationally commutative factor of R is birationally commutative.*

Proof. Since $\ker g = \ker \theta^e$, this is just the universal property of θ^e from Theorem 4.9. \square

7. DEFINING DATA FOR THE CANONICAL BIRATIONALLY COMMUTATIVE FACTOR

We now want to understand the canonical birationally commutative factor $g(R)$ of R . We will see in the next section that $g(R)$ is equal in large degree to a naïve blowup algebra $R(X, \mathcal{L}, \sigma, P)$. In this section, we show that \mathcal{L} is invertible and σ -ample. We construct the 0-dimensional subscheme P that is the missing piece of data for the naïve blowup, and show that $g(R) \subseteq R(X, \mathcal{L}, \sigma, P)$. We show also that the points in P have dense orbits. (In the final section, we prove that they have critically dense orbits, and that the indeterminacy locus of p^{-1} is supported on these orbits.) Throughout, we assume that the hypotheses of Theorem 1.2 hold.

We begin by showing that \mathcal{L} is invertible. We need a further piece of notation.

Notation 7.1. Let R be a connected graded algebra generated in degree 1, and assume that the hypotheses of Theorem 1.2 hold; further adopt Notation 5.8. For $n \geq n_0$ let $X_n \subseteq Y_n$ be the strict transform of X , and let $X_\infty := \varprojlim X_n \subseteq Y_\infty$. Let $W'_n \subseteq W_n$ be the locus in X where $(f_n|_{X_n})^{-1} : X \dashrightarrow X_n$ is undefined, and let $\mathbb{W} := \bigcup_{n \geq n_0} W'_n$.

It is immediate that $\psi_n(X_n) = X_{n-1}$ and $\Psi(X_\infty) = X_\infty$. Thus $X_\infty \subseteq Y_\infty^e$.

Lemma 7.2. *If X is a surface, then X is nonsingular at all points in \mathbb{W} .*

Proof. Let X^{sing} be the singular locus of X . Write $X^{\text{sing}} = X^{(1)} \cup X^{(2)}$, where $X^{(1)}$ is a (possibly reducible or empty) σ -invariant curve and $X^{(2)}$ is 0-dimensional and σ -invariant. Let $w \in \mathbb{W} \cap X^{\text{sing}}$. By Theorem 6.7 the σ -orbit of w is infinite, so $w \in X^{\text{sing}} \setminus X^{(2)} \subseteq X^{(1)}$.

Since $p^{-1} = \Psi p^{-1} \sigma^{-1}$ as rational maps from $X \dashrightarrow X_\infty$, we see that $\sigma^{-1}(\mathbb{W}) \subseteq \mathbb{W}$. As $X^{(1)}$ is σ -invariant,

$$\{\sigma^{-k}(w) \mid k \geq 0\} \subseteq X^{(1)} \cap \mathbb{W},$$

which contradicts Corollary 6.3. Thus no such w can exist. \square

Corollary 7.3. *Adopt Notation 5.8, and assume that the hypotheses of Theorem 1.2 hold. Then \mathcal{L} is invertible, and $g(R) \subseteq B(X, \mathcal{L}, \sigma)$.*

Proof. By Proposition 5.9(5) it suffices to prove that \mathcal{N}_n is invertible for $n \gg 0$. Adopt Notation 7.1. Let $n \geq n_0$, and let \mathcal{R}''_n be the image of the natural map $\mathcal{R}'_n \rightarrow \mathcal{N}_n$. Then \mathcal{R}''_n is the X -torsion-free part of \mathcal{R}'_n . Let $x \in X$. If $x \notin W'_n$, then X_n is locally isomorphic to X at x , and so $(\mathcal{R}''_n)_x = (\mathcal{N}_n)_x$ is invertible.

Now suppose that $x \in W'_n$. If X is locally factorial at points of indeterminacy of p^{-1} , then $\mathcal{N}_n = (\mathcal{R}'_n)^{\vee\vee}$ is invertible by definition. If X is a surface then X is nonsingular at x by Lemma 7.2, and so locally factorial at x . Again $(\mathcal{N}_n)_x$ is invertible.

That $g(R) \subseteq B(X, \mathcal{L}, \sigma)$ follows by construction. \square

Now that we know \mathcal{L} is invertible, we can show that \mathcal{L} is appropriately positive: that is, \mathcal{L} is σ -ample. The condition that \mathcal{L} is σ -ample is technical but important: it means that the twisted tensor powers \mathcal{L}_n have the same good cohomological properties as the powers of an ample invertible sheaf. More formally, we say that a sequence $\{\mathcal{S}_n\}$ of bimodules on a scheme X is *left* (respectively, *right*) *ample* if for every $j \geq 1$ and every coherent sheaf \mathcal{M} on X we have $H^j(X, \mathcal{S}_n \otimes \mathcal{M}) = 0$ (respectively, $H^j(X, \mathcal{M} \otimes \mathcal{S}_n) = 0$) for $n \gg 0$. An invertible sheaf \mathcal{L} is σ -ample if $\{(\mathcal{L}_n)_{\sigma^n}\}$ is left (equivalently, by [Kee00, Theorem 1.2], right) ample.

Let K be the function field of X , and let σ denote also the element of $\text{Aut}_{\mathbb{k}}(K)$ that acts via pullback by $\sigma \in \text{Aut}_{\mathbb{k}}(X)$. Recall that a graded subalgebra S of $K[t; \sigma]$ is *big* if there is some $n \geq 1$ and some $u \in S_n$ so that K is generated by $S_n u^{-1}$ (as a field); morally, we want $K[t, t^{-1}; \sigma]$ to be the graded quotient ring of S .

Theorem 7.4. *Assume that the hypotheses of Theorem 1.2 hold, and further adopt Notation 7.1. Then \mathcal{L} is σ -ample.*

Proof. Fix $n \geq n_0$. Let $f := f_n|_{X_n}$, so f is birational, and let $\mathcal{M} := \mathcal{M}_n|_{X_n}$. There are natural maps

$$H^0(X_n, \mathcal{M}) \otimes \mathcal{O}_{X_n} \longrightarrow f^* f_* \mathcal{M} \longrightarrow \mathcal{M}.$$

Since \mathcal{M} is globally generated, the composition is surjective, and thus $f^* f_* \mathcal{M} \rightarrow \mathcal{M}$ is surjective. It is an isomorphism at the generic point of X_n , and so \mathcal{M} is the torsion-free part of $f^* f_* \mathcal{M}$.

There is a morphism of sheaves $\mathcal{R}'_n \rightarrow f_* \mathcal{M}$ whose kernel and cokernel are supported on W_n . We have $(f_* \mathcal{M})^{\vee\vee} \cong (\mathcal{R}'_n)^{\vee\vee} \cong \mathcal{L}_n$. Pulling back the canonical map $f_* \mathcal{M} \rightarrow (f_* \mathcal{M})^{\vee\vee}$, we obtain a (nonzero) map $f^* f_* \mathcal{M} \rightarrow f^* \mathcal{L}_n$. Since $f^* \mathcal{L}_n$ is invertible and thus torsion-free, by the previous paragraph we obtain an induced injective map $\alpha : \mathcal{M} \rightarrow f^* \mathcal{L}_n$ so that

$$\begin{array}{ccc} f^* f_* \mathcal{M} & \longrightarrow & \mathcal{M} \\ & \searrow & \downarrow \alpha \\ & & f^* \mathcal{L}_n \end{array}$$

commutes. Let $\mathcal{T} := f^* \mathcal{L}_n \otimes \mathcal{M}^{-1}$. Then \mathcal{T} is effective and as a divisor is supported on an f -exceptional divisor, say T .

We claim next that \mathcal{L}_n is ample. We will use terminology and results from [Laz04] on big and nef (Cartier) divisors, and from [Ful98] on intersection theory; in particular, if V is a k -dimensional subvariety of X , let $[V]$ denote the associated k -cycle in $A_k(X)$, the group of k -cycles modulo rational equivalence. Let $f_* : A_k(X_n) \rightarrow A_k(X)$ denote pushforward on cycle classes as in [Ful98]. We identify $A_0(X)$ and $A_0(X_n)$ with \mathbb{Z} ; thus $f_* : A_0(X_n) \rightarrow A_0(X)$ is the identity on \mathbb{Z} .

Let $V \subseteq X$ be a subvariety of dimension $d > 0$. Let \tilde{V} be its strict transform in Y_n . As T is f -exceptional, $\tilde{V} \not\subseteq T$. Thus $\mathcal{T}|_{\tilde{V}}$ is still effective, and as \mathcal{M} is ample, $(f^* \mathcal{L}_n)|_{\tilde{V}} \cong \mathcal{M}|_{\tilde{V}} \otimes \mathcal{T}|_{\tilde{V}}$ is big by [Laz04, Corollary 2.2.7]. (We remark that, although [Laz04] is written with characteristic zero hypotheses throughout, the proof of Corollary 2.2.7 there does not use any hypothesis on the characteristic and the result holds in general.)

We claim that $f^* \mathcal{L}_n$ is nef on X_n , and therefore its restriction is nef on \tilde{V} . To see this, let C be an irreducible curve on X_n . We must show that $c_1(f^* \mathcal{L}_n) \cdot [C] \geq 0$. If f contracts C to a point, then $c_1(f^* \mathcal{L}_n) \cdot [C] = 0$. Otherwise, C is the strict transform of $f(C)$, and by the previous paragraph $f^* \mathcal{L}_n|_C$ is big. It thus has positive degree, and so $c_1(f^* \mathcal{L}_n) \cdot [C] > 0$.

Thus $(f^* \mathcal{L}_n)|_{\tilde{V}}$ is big and nef. By [Laz04, Theorem 2.2.16] (which works in arbitrary characteristic—cf. also [Kee99, Definition-Lemma 0.0]), $\int_{\tilde{V}} (c_1(f^* \mathcal{L}_n))^d > 0$. By repeated applications of the projection formula [Ful98, Proposition 2.5(c)], we obtain

$$\int_V (c_1(\mathcal{L}_n))^d = (c_1(\mathcal{L}_n))^d \cdot f_*[\tilde{V}] = f_*((c_1(f^* \mathcal{L}_n))^d \cdot [\tilde{V}]) = \int_{\tilde{V}} (c_1(f^* \mathcal{L}_n))^d > 0.$$

In summary, we have established that $\int_V (c_1(\mathcal{L}_n))^{\dim V} > 0$ for any irreducible subvariety V of X . The Nakai criterion [Kle66, Theorem III.1] thus implies that \mathcal{L}_n is ample.

It remains to conclude that \mathcal{L} is σ -ample. We write $g(R) \subseteq B(X, \mathcal{L}, \sigma) \subseteq K[t; \sigma]$, and may assume without loss of generality that $t \in g(R_1)$. By Theorem 6.8, we may identify $g(R_n)$ with $\theta^e(R_n)$. This set of sections embeds X_n in some projective space, and as X_n is birational to X the vector space $g(R_n)t^{-n}$ generates K . Thus $g(R)$ is a big subalgebra of $K[t; \sigma]$. As $g(R)$ is noetherian, by [SZ00, Theorem 0.1], $g(R)$ has subexponential growth.

The automorphism σ acts naturally on the vector space $\text{Num}(X)$ of divisors modulo numerical equivalence. By [RZ08, Proposition 3.5], this action must be *quasi-unipotent*: that is, all eigenvalues have modulus 1. By [Kee00, Theorem 1.3], \mathcal{L} is σ -ample. \square

We next prove a lemma on the preimages of points in \mathbb{W} .

Lemma 7.5. *Adopt Notation 7.1 and assume that the hypotheses of Theorem 1.2 hold. Let $n \geq n_0$ and let $f := f_n|_{X_n} : X_n \rightarrow X$. Then $f_*\mathcal{O}_{X_n} = \mathcal{O}_X$. Further, if $w \in W'_n$ then $f^{-1}(w)$ is positive-dimensional.*

Proof. Let $X'_n := \text{Spec}_X f_*\mathcal{O}_{X_n}$ and consider the Stein factorization

$$\begin{array}{ccc} X_n & \xrightarrow{a} & X'_n \\ & \searrow f & \downarrow b \\ & & X \end{array}$$

of f . If X is a surface, by Lemma 7.2 it is normal at all points in W'_n . This also follows if X is locally factorial at points in W_n . In either case the finite map b is a local isomorphism above W'_n , and $f^{-1}(w)$ must be positive-dimensional.

Above points not in W'_n , both a and b are local isomorphisms by definition. Thus b is a local isomorphism at all points and so an isomorphism. \square

We note that this result may be proved without the local factoriality assumption on X , merely using Theorem 6.7. We do not include this proof here.

We now define the final piece of data we need to define the naïve blowup algebra $R(X, \mathcal{L}, \sigma, P)$. This will allow us to show that \mathbb{W} is supported on finitely many dense σ -orbits.

Notation 7.6. Assume that the hypotheses of Theorem 1.2 hold, and adopt Notation 7.1. For $n \in \mathbb{N}$, let $\mathcal{R}_n \subseteq \mathcal{L}_n$ be the subsheaf generated by the sections in $g(R_n)$. Let P be the base locus of $g(R_1)$ — that is, define P by $\mathcal{I}_P = \mathcal{R}_1\mathcal{L}^{-1}$.

Lemma 7.7. *Assume that the hypotheses of Theorem 1.2 hold, and adopt Notation 7.1.*

- (1) *We have $g(R) \subseteq R(X, \mathcal{L}, \sigma, P)$.*
- (2) *Given $n \in \mathbb{N}$, define*

$$P_n := \text{Supp}(P \cup \sigma^{-1}(P) \cup \dots \cup \sigma^{-(n-1)}(P)).$$

Then $P_n = W'_n$ for $n \geq n_0$, and set-theoretically we have $\mathbb{W} = \bigcup_{k \geq 0} \sigma^{-k}(P)$.

Proof. Since \mathcal{R}_n is generated by $g(R_n) = g(R_1)^n$, we have $\mathcal{R}_n = \mathcal{R}_1\mathcal{R}_1^\sigma \dots \mathcal{R}_1^{\sigma^{n-1}}$. (1) follows immediately.

(2). We see that \mathcal{R}_n is invertible at x if and only if each of the $\mathcal{R}_1^{\sigma^i}$ are invertible at x , for $0 \leq i \leq n-1$: that is, P_n is exactly the locus where \mathcal{R}_n is not invertible.

Let $f := f_n|_{X_n}$. Let $w \in W'_n$ and let $T := f^{-1}(w) \subseteq X_n$. By Lemma 7.5, $\dim T \geq 1$. Let $r \in R_n \setminus \ker \theta'$. Then $\theta'(r)$ is a nonzero element of $H^0(Y'_n, \mathcal{M}'_n)$. Because $\dim T \geq 1$ and \mathcal{M}'_n is very ample, $\theta'(r)$ must vanish at some point of T . Thus $g(r)$, regarded as a section of the invertible sheaf \mathcal{L}_n , must vanish at w and is contained in $H^0(X, \mathcal{I}_w\mathcal{L}_n)$. We see that $\mathcal{R}_n \subseteq \mathcal{I}_w\mathcal{L}_n$, and $w \in P_n$.

Now suppose that $x \notin W'_n$. Let \mathcal{R}''_n be the image of the natural map $\mathcal{R}'_n \rightarrow \mathcal{L}_n$. Since f is a local isomorphism above x , therefore $(\mathcal{R}''_n)_x = (\mathcal{L}_n)_x$ is invertible.

The sections in (the image of) R_n generate $\mathcal{M}_n|_{X_n}$ at all points of X_n . Thus their images under g must locally generate \mathcal{R}''_n at x , and so

$$(\mathcal{R}_n)_x = (\mathcal{R}''_n)_x = (\mathcal{L}_n)_x,$$

and $x \notin P_n$.

Thus $P_n = W'_n$. It follows that $\mathbb{W} = \bigcup_{k \geq 0} \sigma^{-k}(P)$. \square

Our next goal is to show that all points of P have dense orbits. Before proving this result, we need a lemma.

Lemma 7.8. *Let X be a projective scheme of dimension ≥ 2 , let $\sigma \in \text{Aut}_{\mathbb{K}} X$, and let \mathcal{L} be a σ -ample invertible sheaf on X . Let \mathcal{I} be an ideal sheaf on X , and let P be the subscheme defined by \mathcal{I} . For $n \geq 0$, let $\mathcal{S}_n := \mathcal{I}\mathcal{I}^\sigma \dots \mathcal{I}^{\sigma^{n-1}}\mathcal{L}_n$.*

Suppose that (1) $\dim P = 0$ and (2) for every $x \in \text{Supp } P$ and every irreducible component Y of X so that $\{\sigma^n(x)\}$ meets Y , the set $\{\sigma^n(x)\} \cap Y$ is Zariski-dense in Y . Then $\{(\mathcal{S}_n)_{\sigma^n}\}$ is a left and right ample sequence on X .

Proof. By symmetry, it suffices to prove that the sequence is right ample. By replacing σ by a power, we may suppose that each irreducible component of X is σ -invariant.

Let $X^{(i)}$ denote the i th infinitesimal neighborhood of $X_{\text{red}} \subseteq X$; then $X^{(N)} = X$ for some N . We first reduce to showing that $\{\mathcal{S}_n|_{X_{\text{red}}}\}$ is an ample sequence. Let \mathcal{M} be a coherent sheaf supported on $X^{(m)}$. Then there is an exact sequence

$$0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0,$$

where \mathcal{M}'' is supported on X^{red} and \mathcal{M}' is supported on $X^{(m-1)}$. Fix $j \geq 1$. For any n , there is an exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{M}' \otimes \mathcal{S}_n \rightarrow \mathcal{M} \otimes \mathcal{S}_n \rightarrow \mathcal{M}'' \otimes \mathcal{S}_n \rightarrow 0,$$

where \mathcal{K} is supported on a set of dimension 0. Assume, by way of induction hypothesis, that $H^j(X, \mathcal{M}' \otimes \mathcal{S}_n)$ and $H^j(X, \mathcal{M}'' \otimes \mathcal{S}_n)$ vanish for $j \geq 1$ and $n \gg 0$. Then $H^j(X, \mathcal{M} \otimes \mathcal{S}_n) = 0$ for $j \geq 1$ and $n \gg 0$. Taking $m = N$, we conclude that if $\{\mathcal{S}_n|_{X_{\text{red}}}\}$ is an ample sequence then so is $\{\mathcal{S}_n\}$.

Now suppose that $X = X_{\text{red}}$ is reduced and let $\{X_c\}_{c=1, \dots, \ell}$ be a list of the irreducible components of X . Suppose that \mathcal{M} is a coherent sheaf supported on $X_1 \cup \dots \cup X_j$ (for some $1 \leq j \leq \ell$). We have an exact sequence

$$0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}|_{X_j} \rightarrow 0$$

with \mathcal{M}' supported on $X_1 \cup \dots \cup X_{j-1}$. As all points of P have dense orbits in their components, no translate of any point in P can lie in the intersection of two components. Thus the sequence

$$0 \rightarrow \mathcal{M}' \otimes \mathcal{S}_n \rightarrow \mathcal{M} \otimes \mathcal{S}_n \rightarrow \mathcal{M}|_{X_j} \otimes \mathcal{S}_n \rightarrow 0,$$

obtained by tensoring with \mathcal{S}_n remains exact. An induction thus reduces us to showing that $\{\mathcal{S}_n|_{X_j}\}$ is a right ample sequence for each j : that is, it suffices to prove the result in the case that X is both irreducible and reduced. But this is precisely [RS07, Theorem 3.1(1)]. \square

Proposition 7.9. *Assume that the hypotheses of Theorem 1.2 hold, and further adopt Notation 7.6. Then any $w \in P$ has a dense orbit in X .*

Proof. Suppose that for some $w \in P$, the orbit closure $\Gamma := \overline{\{\sigma^n(w)\}} \neq X$. Now Γ is σ -invariant, and by passing to a Veronese and replacing σ by a power, we may assume that Γ is irreducible. We may also assume that Γ is minimal: i.e. any $y \in P \cap \Gamma$ has a dense orbit in Γ . By Corollary 6.3, $\dim \Gamma \geq 2$.

Since the cokernel of the maps $\mathcal{R}_n \rightarrow \mathcal{L}_n$ is 0-dimensional, and all points in P and W'_n have infinite σ -orbits, it is an easy exercise that there is some $k \in \mathbb{N}$ so that

$$(7.10) \quad (\mathcal{R}_m)_x \supseteq (\mathcal{I}_\Gamma^{(k)} \mathcal{L}_m)_x$$

for all $x \in \Gamma$ and for all $m \geq n_0 \in \mathbb{N}$. (Here $\mathcal{I}_\Gamma^{(k)}$ is the k 'th symbolic power of \mathcal{I}_Γ ; we refer to the subscheme of X it defines as $k\Gamma$.) By increasing k , we may assume (7.10) holds for all $m \in \mathbb{N}$.

Let $\Gamma' := (k+1)\Gamma$ be the subscheme of X defined by $\mathcal{I}_\Gamma^{(k+1)}$. Let $\mathcal{F} := \mathcal{L}|_{\Gamma'}$. By Theorem 7.4, \mathcal{F} is σ -ample on Γ' .

For each n , let $\mathcal{S}_n \subseteq \mathcal{F}_n$ be the image of \mathcal{R}_n in \mathcal{F}_n . Since \mathcal{F}_n is invertible, we may define $\mathcal{J}_n := \mathcal{S}_n(\mathcal{F}_n)^{-1} \subseteq \mathcal{O}_{\Gamma'}$. We have $\mathcal{J}_n = \mathcal{J}_1 \mathcal{J}_1^\sigma \cdots \mathcal{J}_1^{\sigma^{n-1}}$.

Let $S := \bigoplus H^0(\Gamma', \mathcal{S}_n)$. Since the points in the subscheme defined by \mathcal{J}_n have dense orbits in Γ , by Lemma 7.8 the sequence of bimodules $\{(\mathcal{S}_n)_{\sigma^n}\}$ is left and right ample on Γ' . By [Sie11, Lemma 7.4], S is a finitely generated left and right module over R . Thus S is left and right noetherian.

Let \mathcal{H} be the ideal sheaf on Γ' defining $k\Gamma \subseteq \Gamma'$. We have $\mathcal{J}_n \supseteq \mathcal{H}$ for all $n \in \mathbb{N}$, by (7.10). Let

$$H := \bigoplus_{n \geq 0} H^0(\Gamma', \mathcal{H}\mathcal{F}_n).$$

Since $\mathcal{H}\mathcal{F}_n \subseteq \mathcal{S}_n$ for all n and $k\Gamma$ is σ -invariant, this is a two-sided ideal of S .

By Lemma 7.7, $\mathcal{J}_n \subseteq \mathcal{I}_w$ for all $n \in \mathbb{N}$. Let $m \geq 1 \in \mathbb{N}$. It follows from Nakayama's lemma that $(\mathcal{H} \cap \mathcal{J}_n) \mathcal{J}_m^{\sigma^n} = \mathcal{H} \mathcal{J}_m^{\sigma^n} \subsetneq \mathcal{H} = \mathcal{H} \cap \mathcal{J}_{n+m}$ for any $n \geq 1$. Since \mathcal{F} is σ -ample, for $n \gg 0$ the sheaf $\mathcal{H}\mathcal{F}_{n+m}$ is globally generated. Thus for $n \gg 0$, we have $H_n \cdot S_m \subseteq H^0(\Gamma', (\mathcal{H}\mathcal{F}_n) \mathcal{S}_m^{\sigma^n}) \subsetneq H_{n+m}$. This shows H is not finitely generated as a right ideal, giving a contradiction.

Thus the orbit of w is dense in X . \square

8. THE MAIN THEOREM

In this section, we prove Theorem 1.2; in fact, we prove a stronger structure theorem for R and for the point space Y_∞ .

Theorem 8.1. *Let R be a noetherian connected graded \mathbb{k} -algebra generated in degree 1, where \mathbb{k} is an algebraically closed uncountable field. Let Y_∞ be the point space of R , and let F, G be as in Notation 2.4. Suppose:*

- (i) *there is a commutative diagram*

$$\begin{array}{ccc} & F \cong Y_\infty & \\ \pi \swarrow & & \searrow p \\ G & \xrightarrow{\quad} & X \end{array}$$

where X is a projective scheme and a coarse moduli space for G .

Suppose further that:

- (ii) X is a variety of dimension ≥ 2 that is either a surface or locally factorial at all indeterminacy points of p^{-1} ;
- (iii) the map $G \rightarrow X$ is bijective on \mathbb{k} -points;
- (iv) the indeterminacy locus of p^{-1} consists (set-theoretically) of countably many points;

Then:

- (a) Y_∞ is noetherian;
- (b) Let Ω be the indeterminacy locus of p^{-1} . Then there are an automorphism σ of X and a finite subset P' of X , all of whose points have critically dense orbits, so that Ω is supported on half-orbits of points in P' . In particular, Ω is critically dense in X .
- (c) there are a σ -ample invertible sheaf \mathcal{L} on X and a closed subscheme P of X with $\text{Supp } P = P'$ so that there is a homomorphism, surjective in large degree,

$$g : R \rightarrow R(X, \mathcal{L}, \sigma, P).$$

Furthermore:

- (d) any graded homomorphism from R to a birationally commutative algebra factors through g in large degree.
- (e) Define θ', θ^e as in Section 4. Then $\ker g = \ker \theta' = \ker \theta^e$.

Proof. (e). Let $K := \mathbb{k}(X)$. Let $\sigma \in \text{Aut}_{\mathbb{k}}(X)$ and $g : R \rightarrow K[t; \sigma]$ be given by Theorem 6.8. We write $S := g(R)$. By Theorem 6.8, we have $\ker g = \ker \theta' = \ker \theta^e$.

(d) is the universal property of θ^e in Theorem 4.9.

(c) Let the rank 1 reflexive sheaf \mathcal{L} on X be as in Notation 5.8. By Corollary 7.3, \mathcal{L} is invertible and $S \subseteq B(X, \mathcal{L}, \sigma)$. By Theorem 7.4, \mathcal{L} is σ -ample.

For $n \geq 0$ define $\mathcal{R}_n \subseteq \mathcal{L}_n$ to be the subsheaf generated by the sections in $g(R_n)$. Let P be the subscheme of X defined by $\mathcal{R}_1 \mathcal{L}^{-1}$. Let $P' := \text{Supp } P$. Since $g(R)$ is generated in degree 1, we have $\mathcal{R}_n = \mathcal{R}_1 \mathcal{R}_1^\sigma \cdots \mathcal{R}_1^{\sigma^n}$, and $g(R) \subseteq R(X, \mathcal{L}, \sigma, P) \subseteq B(X, \mathcal{L}, \sigma)$.

Adopt Notation 7.1. That is, for $n \geq n_0$, let $X_n \subseteq Y'_n$ be the strict transform of X . Recall that W'_n is the indeterminacy locus of the rational map $(f_n|_{X_n})^{-1} : X \dashrightarrow X_n$. By Lemma 7.7, we have $W'_n = P' \cup \cdots \cup \sigma^{-(n-1)}(P')$ for $n \geq n_0$. By Proposition 7.9, all points in P' have dense orbits.

The sections in $g(R_n) = \theta'(R_n)$ define an immersion of $X \setminus W'_n \cong f_n^{-1}(X \setminus W'_n) \cap X_n \hookrightarrow \mathbb{P}^d$ for appropriate d . We apply the following result of Rogalski and Stafford:

Theorem 8.2. ([RS07, Theorem 9.2]) *Let X be a projective variety, let $\sigma \in \text{Aut}(X)$, and let \mathcal{L} be a σ -ample invertible sheaf on X . Let $\mathcal{I} = \mathcal{I}_P$ be an ideal sheaf on X so that P is 0-dimensional and so that all points in P have dense orbits.*

Let $S' \subseteq R(X, \mathcal{L}, \sigma, P)$ be a noetherian subalgebra so that

- (1) *for $n \gg 0$ the sections in S'_n generate $\mathcal{I}_P(\mathcal{I}_P)^\sigma \cdots (\mathcal{I}_P)^{\sigma^{n-1}} \mathcal{L}_n$; and*
- (2) *for $n \gg 0$ the sections in S'_n restrict to give an immersion*

$$X \setminus (P \cup \sigma^{-1}(P) \cup \cdots \cup \sigma^{-(n-1)}(P)) \hookrightarrow \mathbb{P}^N.$$

Then $S' = R(X, \mathcal{L}, \sigma, P)$ up to finite dimension, and all points in P have critically dense orbits.

We obtain immediately that $g(R)$ and $R(X, \mathcal{L}, \sigma, P)$ are equal in large degree, and that all points in P' have critically dense orbits.

(a). By part (e), $\ker g$ annihilates any R -point module. Thus Y_∞ is also the point space for S , and is noetherian by Theorem 3.3.

(b) Let $q : Y_\infty \rightarrow X$ be the map constructed in Proposition 3.4. Since q is constant on \sim -equivalence classes, it factors through p , and there is a morphism $\tau : X \rightarrow X$ so that $\tau p = q$. By Proposition 3.4(iii), q^{-1} is defined at the generic point of X . As p^{-1} is defined at the generic point of X by assumption, τ is birational and thus:

Claim 8.3. τ is an automorphism of X .

Proof of Claim. As in the proof of Lemma 2.3(1) of [Fuj02], τ is finite, as well as birational. (We note the cited result applies to projective nonsingular varieties defined over \mathbb{C} ; but the latter two hypotheses are not used in the proof we cite.) By finiteness of the integral closure, $(\tau^n)_* \mathcal{O}_X = (\tau^{n+1})_* \mathcal{O}_X$ for some $n \in \mathbb{N}$. As τ^n is finite and so affine, we have $\mathcal{O}_X = \tau_* \mathcal{O}_X$, and so τ is an isomorphism. \square

Returning to the proof of the theorem, let Ω be the indeterminacy locus of p^{-1} ; then $\tau(\Omega)$ is the indeterminacy locus of q^{-1} . By Proposition 3.4(iii), we have

$$\tau(\Omega) = \bigcup \{\sigma^{-n}(P') \mid n \geq 0\}.$$

Consider the maps

$$\begin{array}{ccccc} X & & \xrightarrow{\tau} & & X \\ & \swarrow p & & \searrow q & \\ & Y_\infty & & & \\ & \swarrow p & & \searrow q & \\ X & & \xrightarrow{\tau} & & X \\ & \swarrow p & & \searrow q & \\ & Y_\infty & & & \end{array}$$

The top and bottom faces of the diagram commute by definition of τ , and the left face commutes by the construction of σ in Proposition 5.1. The right face commutes by Proposition 3.4(†). Thus

$$\sigma \tau p = \sigma q = q \Psi = \tau p \Psi = \tau \sigma p.$$

By Lemma 2.2 we have

$$(8.4) \quad \sigma \tau = \tau \sigma.$$

Note that p^{-1} is not defined at any point in $\bigcup_n W'_n = \bigcup \{\sigma^{-n}(P') \mid n \geq 0\}$. We thus have

$$\bigcup \{\sigma^{-n}(P') \mid n \geq 0\} \subseteq \Omega = \bigcup \{\tau^{-1} \sigma^{-n}(P') \mid n \geq 0\} = \bigcup \{\sigma^{-n} \tau^{-1}(P') \mid n \geq 0\},$$

by (8.4). Thus τ permutes the finitely many orbits of points in P' , and thus there is some $k < 0$ so that $\bigcup \{\sigma^{-n} \tau^{-1}(P') \mid n \geq 0\} \subseteq \bigcup \{\sigma^{-n}(P') \mid n \geq k\}$. Thus

$$\bigcup \{\sigma^{-n} \tau^{-1}(P') \mid n \geq 0\} \subseteq \Omega \subseteq \bigcup \{\sigma^{-n}(P') \mid n \geq k\}.$$

We observed already that all points in P' have critically dense orbits.¹ \square

Theorem 1.2 is immediate from Theorem 8.1. We note that we do not believe that the local factoriality hypothesis in Theorem 1.2 is necessary.

To end, we comment on some open problems. The results in this paper naturally lead to the question of understanding the canonical birationally commutative factors of noetherian algebras: that is, of classifying noetherian birationally commutative graded algebras generated in degree 1. We believe that Theorem 8.1 should have an extension to any situation where there is a coarse moduli scheme for tails of point modules. In particular, we conjecture that given a noetherian connected graded \mathbb{k} -algebra R , generated in degree 1, and a projective scheme X that is a coarse moduli space for tails of R -points, the image of R inside $\mathbb{k}(X)[t; \sigma]$ is in fact contained in the twisted homogeneous coordinate ring of a σ -ample invertible sheaf on X . We cannot prove this, however; moreover, understanding the image of R would require understanding the analogues of naïve blowup algebras at positive-dimensional subschemes, and these have not yet been studied.

¹In fact, for appropriate m, n we have $\sigma^n = \tau^m$. To see this, let $m \neq 0$ be such that τ^m leaves some orbit of a point in P invariant. Then for some $p \in P$ and $n \in \mathbb{Z}$, we have $\tau^m(p) = \sigma^n(p)$. Since the fixed point set of $\sigma^n \tau^{-m}$ is Zariski-closed in X and includes all points in the (dense) orbit of P , it must consist of all of X .

More interesting, perhaps, is the existence of noetherian birationally commutative algebras with no coarse moduli scheme of tails of points. The algebra discussed in [RS11] is of this type, as Rogalski and the second author will show in forthcoming work. Thus algebras such as those appearing in [RS11] must appear in any putative classification of canonical birationally commutative factors. Note this algebra has no nonzero maps to a twisted homogeneous coordinate ring! We cannot yet precisely define what geometry characterizes such algebras; these questions are the subject of ongoing research.

Finally, we note that if P is supported on infinite but not critically dense σ -orbits, then $R(X, \mathcal{L}, \sigma, P)$ is not noetherian and neither (given sufficient ampleness of \mathcal{L}) is its point space. This contrasts with the conclusion of Theorem 1.2 that Y_∞ is noetherian. Thus the results here give some evidence for the point of view that noetherianness can be detected geometrically. We conjecture that the point space Y_∞ of any noetherian algebra is noetherian.

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